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Splitting Frequencies for Resonant Solutions in Central Fields

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Abstract

The behavior of a mass point moving in a plane under the effect of a central field and an external periodic excitation in resonance with the natural frequency is studied. The asymptotic perturbation method is used in order to determine the nonlinear modulation equations for the amplitude and the phase of the oscillation. Firstly, we calculate the second order approximate solution of the unforced system. It is well known that generally the solution is two period quasiperiodic, but we find some new cases of periodic solutions. If appropriate Diophantine equations are satisfied, the motion is periodic with a frequency depending on the nonlinear terms. Subsequently, the forced system is considered and external force-response curves are shown and moreover jump phenomena are also observed. In certain cases we observe a frequency splitting and a third frequency appears in addition to the forcing frequency Stable three period quasi-periodic motions are present with amplitudes depending on the initial conditions.

Keywords: central field, quasi-periodic motion, nonlinear oscillations, perturbation methods

1. Introduction

We study the behavior of a pointlike body moving on a plane under the effect of a central field and a weak external periodic excitation in resonance or in quasi-resonance with the natural frequency of the system. The unforced system is described by the Lagrangian

$$\mathrm{L}=\frac{1}{2}m\left((\dot{R})^2+\mathrm{R}^2\left(\dot{\theta}\right)^2\right)-f(R), \tag{1}$$

where R and $\vartheta$are usual polar coordinates, *m* is the mass and *f(R)* the central field potential. The Lagrange equations are

$$m\ddot{R}-mR\dot{\theta}^2+f'(R)=0, \tag{2}$$

$$\mathrm{mR}^2\dot{\theta}=constant=K. \tag{3}$$

Using equation (3), we can write equation (2) in the following form

$$m\ddot{R}=\mathrm{g}(R)-\mathrm{f}'(R) \tag{4}$$

where

$$g(R) = \frac{K^2}{mR^3}. \tag{5}$$

We consider the motion near the equilibrium point R_0 defined by

$$f'(R_0) = g(R_0) = \frac{K^2}{mR_0^3}, \tag{6}$$

and with a Taylor series up to cubic terms we obtain ($R \rightarrow R_0$+R)

$$m\ddot{R} = \left(g'(R_0) - f''(R_0)\right)(R) + \frac{\left(g''(R_0) - f'''(R_0)\right)}{2}(R)^2 + \frac{g'''(R_0) - f^{iv}(R_0)}{6}(R)^3 \tag{7}$$

where

$$g'(R_0) = -\frac{3K^2}{mR_0^4},\ g''(R_0) = \frac{12K^2}{mR_0^5},\ \ g'''(R_0) = -\frac{60K^2}{mR_0^6}. \tag{8}$$

We have neglected powers superior to the third because they do not affect the second order approximate solution (see Section 2).

We now suppose that the oscillator is under the effect of an external excitation,

A fundamental resonance is verified if $\Omega \sim \omega$, where ω is the natural frequency of the circular orbit ($R=R_0$) and is given by

$$\omega^2 = \frac{1}{m}\left(f''(R_0) + \frac{3K^2}{mR_0^4}\right). \tag{9}$$

Equation (7) can be rewritten as

$$\ddot{R} = \omega^2 R + bR^2 + cR^3 + 2F\cos(\Omega t) \tag{10}$$

where

$$b = \frac{1}{2m}\left(\frac{12K^2}{mR_0^5} - f'''(R_0)\right),\ \ c = -\frac{1}{6m}\left(\frac{60K^2}{mR_0^6} + f^{IV}(R_0)\right),\qquad F = \frac{F}{m}. \tag{11}$$

A very simple method to construct approximate solutions of the equation (10) is the harmonic balance method [1-2], which considers the solution to be a sum of several harmonics. As it is well known, the trouble is that, beyond the very rough approximation of the first one or two harmonics, the algebra becomes horrendous very quickly and there is no systematic way to obtain improved solutions.

Analytical solutions can be also obtained by means of various perturbation techniques, for example averaging procedure and multiple scales method [3-6].

In this paper we use the asymptotic perturbation (AP) method [7-10] and study its ability to construct a quite satisfactory approximate solution.

The AP method provides a systematic means to obtain increasingly accurate solution by increasing the order of approximation in terms of a small parameter □. This is accomplished by three processes: obtaining the form of solution in term of harmonic components, introducing a large temporal rescaling and solving directly for the various harmonic components via harmonic balance.

In Section 2 we calculate the second order approximate solution of the unforced system and demonstrate that if appropriate Diophantine equations are satisfied, the motion is periodic with a frequency depending on the nonlinear terms.

In Section 3 we consider the forced system and show that under appropriate conditions a frequency splitting occurs and the motion is three period quasi-periodic, with three frequencies, because a third frequency appears and then stable quasi-periodic motions are present with amplitudes depending on the initial conditions. The value of the low frequency depends on the amplitude of the external excitation. Moreover, jump phenomena are also observed in the external force-response curve, because in certain cases and for increasing external excitation the amplitude of the oscillation suddenly can pass from low to high values.

In the last section, we summarize the most important results and indicate some possible extensions and generalizations.

2. The Approximate Second Order Solution for the Unforced System

We introduce the small positive nondimensional parameter ε to serve as a bookkeeping device; it will be set equal to unity in the final analysis. We rewrite equation (10) with *F=0* in the following form

$$\ddot{R}=\omega^2 R+bR^2+cR^3 \tag{12}$$

and then we introduce a temporal rescaling

$$t=e^{q}t, \quad q>0, \tag{13}$$

where q is a rational number to be fixed later on.

If we take e=0 in equation (12), nonlinear terms vanish and we see that it admits simple harmonic solutions $X(t)$=Aexp(−it)+c.c., where *A* is a constant depending on initial conditions and $c.c.$ stands for complex conjugate. Nonlinear effects induce a modulation of the amplitude *A* and the appearance of higher harmonics. The modulation is best described in terms of the rescaled variable τ, that accounts for the need to look on larger time scales, to obtain a significant contribution from nonlinear terms.

The solution *R(t)* of Equation (12) can be expressed by means of a power series in the expansion parameter □ as

$$X(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\gamma_n}\psi_n(\tau,|\varepsilon)\exp(-in\omega t), \tag{14}$$

where $g_n = |n|$-*1* for $n^{1}0$, g_0=r is a non-negative number, which will be fixed later on, and y_n(t,e)=y $^{-n}$(t,e), because *R(t)* is real. The assumed solution (14) can be considered as a combination of the different harmonics, solutions of the linear equation. The coefficients of this combination depend on □ and □. Equation (14) can be written more explicitly

$$R(t)=\varepsilon^r\psi_0(\tau;\varepsilon) + \left(\psi_1(\tau;\varepsilon)\exp(-i\omega t)+\varepsilon\psi_2(\tau;\varepsilon)\exp(-2i\omega t)+c.c.\right)+o(\varepsilon^2). \tag{15}$$

Moreover the variable change (13) implies

$$\frac{d}{dt}\left(\psi_n\exp(-in\omega t)\right) = \left(-in\omega+\varepsilon^q\frac{d\psi_n}{d\tau}\right)\exp(-in\omega t). \tag{16}$$

The functions y_n(t,e)'s depend on the parameter □ and we suppose that the limit of the y_n's for $e \to 0$ exists and is finite and they can be expanded in power series of ε, i.e.

$$y_n(t;e) = \sum_{i=0}^{¥} e^i\, y_n^{(i)}(t). \tag{17}$$

In the following for simplicity we use the abbreviations $y_n^{(0)}$=y$_n$ for n¹1 and $y_1^{(0)}$=y for n=1. The expansion of the solution, we have chosen in Equation (14), is used for the elimination of the predominant linear part of the Equation (12) and it allows to calculate the possible interactions among the different harmonics, created by the nonlinear terms. The expansion (14) can be substituted into the equation (12) to obtain separate equations for each *n* and for a fixed order of approximation. We can try and obtain that only y_1=y appears in our equations, because every y_n can be expressed by means of it.

Considering the equations for *n=2* and *n=0*, we obtain

$$\varepsilon\psi_2 = -\varepsilon\left(\frac{b}{3\omega^2}\right)\psi^2 + o(\varepsilon), \tag{18}$$

$$\varepsilon^r\psi_0 = 2\varepsilon\frac{b}{\omega^2}|\psi|^2 + o(\varepsilon). \tag{19}$$

From Equation (19), we see that the correct magnitude order of y_0 is r=1, while, for *n=1*, Equation (12) yields

$$2i\omega\varepsilon^q\psi_\tau + 2b\varepsilon\left(\psi_0\psi + \psi_2\psi\right) + 3c\varepsilon^2|\psi|^2\psi = 0. \tag{20}$$

The choice *q=2* is requested for the proper balance of terms. It reveals the temporal scale, when the nonlinear contributions can modulate the solution amplitude.

After substituting Equations (18-19) into Equation (20), we obtain

$$2i\omega\varepsilon^q\psi_\tau + \left(\frac{10b^2}{3\omega^2} + 3c\right)\varepsilon^2|\psi|^2\psi = 0. \tag{21}$$

Via the transformation

$$\psi(\tau) = \rho(\tau)\exp\left(i\alpha(\tau)\right), \tag{22}$$

we arrive at the model equations

$$\frac{d\rho}{d\tau} = 0, \quad \frac{d\alpha}{d\tau} = N\rho^2, \tag{23}$$

where

$$N = \frac{1}{2\omega}\left(\frac{10b^2}{3\omega^2} + 3c\right). \tag{24}$$

From (3), we get at the second order

$$\frac{K}{mR^2} = \omega_0\left(1 - 2\frac{R}{R_0} + 3\frac{R^2}{R_0^2}\right). \tag{25}$$

Where $$\omega_0 = \frac{K}{mR_0^2}\,. \tag{26}$$

From (15), (23) and (25), we obtain the second order approximate solution,

$$R(t) = R_0 + 2\rho_I\cos(\omega_2 t - \alpha) + \frac{2b}{\omega_2^2}\rho_I^2 - \frac{2b}{3\omega_2^2}\rho_I^2\cos(2\omega_2 t - 2\alpha), \tag{27}$$

$$\begin{gathered}\vartheta(t) = \vartheta_I + \omega_1 t - \frac{4\omega_1\rho_I}{R_0\omega_2}\left(\sin(\omega_2 t - \alpha_I) + \sin\alpha_I\right)\\ \frac{+\omega_1\rho_I^2}{\omega_2}\left(\frac{3}{R_0^2} + \frac{2b}{\omega_2^2}\right)\left(\sin(2\omega_2 t - 2\alpha_I) + \sin 2\alpha_I\right)\end{gathered}, \tag{28}$$

Where $$\omega_1 = \omega_0 + \varpi_0,\ \varpi_0 = \frac{2\omega_0\rho_I^2}{R_0}\left(\frac{3}{R_0} - \frac{2b}{\omega_2^2}\right), \tag{29}$$

$$\omega_2 = \omega + ,\varpi = N\rho_I^2. \tag{30}$$

and ρ_I, α_I, ϑ_I are fixed by the initial conditions.

The second order approximate solution in Cartesian coordinates is

$$X(t)=\mathrm{R}(t)\cos\big(\vartheta(t)\big), \tag{31}$$

$$Y(t)=\mathrm{R}(t)\sin\big(\vartheta(t)\big), \tag{32}$$

where *R(t)* is given by (27) and$\vartheta(t)$ by (28).

In general, we observe a two period quasi periodic motion with the incommensurable frequencies ω_1 and ω_2. However, from equations (27) and (28) we can see that the motion can be periodic if the two frequencies are commensurable, i.e.

$$\omega_2 = \frac{p}{q}\omega_1, \quad . \tag{33}$$

where *p* and *q* are integers. As consequence,

$$\frac{\omega_2}{\omega_1} = \frac{s_2}{s_1}, \tag{34}$$

where s_2 and s_1 are integers. We then consider the Diophantine equation

$$s_1 p - s_2 \mathrm{q}=0. \tag{35}$$

The general solution is

$$\mathrm{q}=\mathrm{s}_1 t, \quad \mathrm{p}=\mathrm{s}_1 t\ , \tag{36}$$

where *t* is an integer. The equation (33) yields at the first order

$$\omega_0^2 = \frac{s_1^2 f''(R_0)}{m(s_2^2-3s_1^2)}, \tag{37}$$

for the periodic solutions of the system (2-3).

As example, we consider the potential

$$f(R) = \frac{\beta}{R} + \frac{\delta}{2}R^2, \qquad \delta>0 \tag{38}$$

and with the condition (34), we obtain at the first order of approximation a quadratic Diophantine equation

$$(x-4)s_1^2 + (1-x)s_2^2 = 0\ , \qquad \mathrm{x}=\frac{\beta}{\delta \mathrm{R}_0^3}, \tag{39}$$

or

$$\mathrm{x}=\frac{4s_1^2-s_2^2}{s_1^2-s_2^2}. \tag{40}$$

We can easily calculate *x* for some values of s_1 and s_2

$$s_1 = 1,\ s_2 = 3,4,5....\quad \mathrm{x}=\frac{5}{8},\frac{4}{5},\frac{7}{8}.... \tag{41}$$

$$s_1 = 1,\ s_2 = 1,3,5....\quad \mathrm{x}=5,-\frac{5}{5},\frac{3}{7}.... \tag{42}$$

$$s_1 = 3,\ s_2 = 1,2,4,5....\quad \mathrm{x}=\frac{35}{8},\frac{32}{5},-\frac{20}{7},-\frac{11}{16}.... \tag{43}$$

$$s_1 = 4,\ s_2 = 3,4,5....\quad \mathrm{x}=\frac{63}{15},5,\frac{55}{7},-\frac{13}{3}... \tag{44}$$

$$s_1 = 4,\ s_2 = 3,4,5....\quad \mathrm{x}=\frac{33}{8},\frac{32}{7},\frac{91}{16},\frac{28}{3}... \tag{45}$$

Note that negative values of the ratio *x* imply that the parameter β of the potential (38) is negative (Coulombian attractive potential). In conclusion, we have demonstrated that only for appropriate rational values of the ratio x (40) the solution of the system (2-3) with the potential (38) is periodic.

In Fig. 1 we show a periodic motion in the case $\omega_2 = \frac{3}{2}\omega_1$, x$= -\frac{7}{5}$ and in Fig. 2 in the case $\omega_2 = \frac{4}{3}\omega_1$, x$= -\frac{20}{7}$.

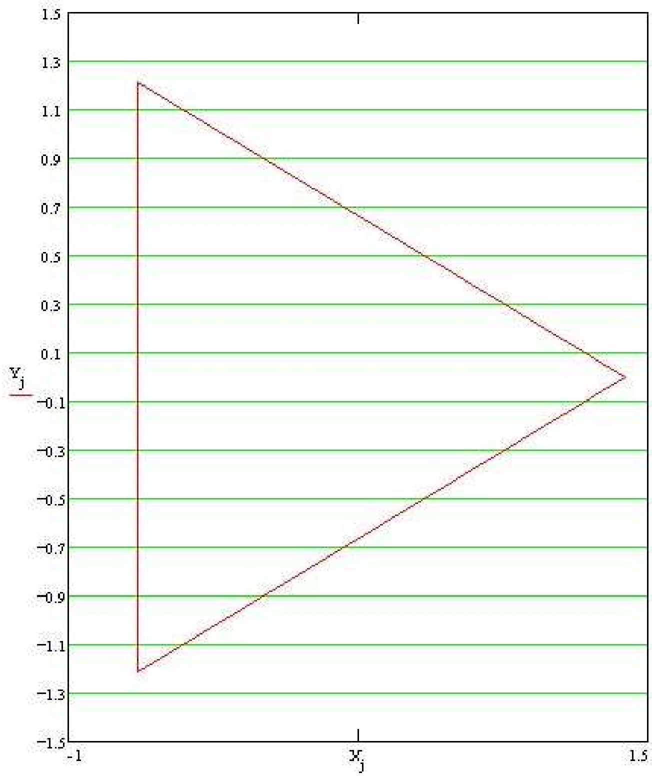

Figure 1: Periodic motion in the case ω$=\frac{3}{2}\omega_1$. It is represented in the plane X-Y the associated map, i.e. the values of the approximate solution for time t$=\frac{2\pi}{\omega_2}n$, where n is a positive integer.

In particular, it is represented in the plane X-Y the associated map, i.e. the values of the approximate solution given by (31-32) for time t$=\frac{2\pi}{\omega_2}n$, where n is a positive integer. The solution is periodic with period T$=\frac{2\pi}{\omega_1}$.

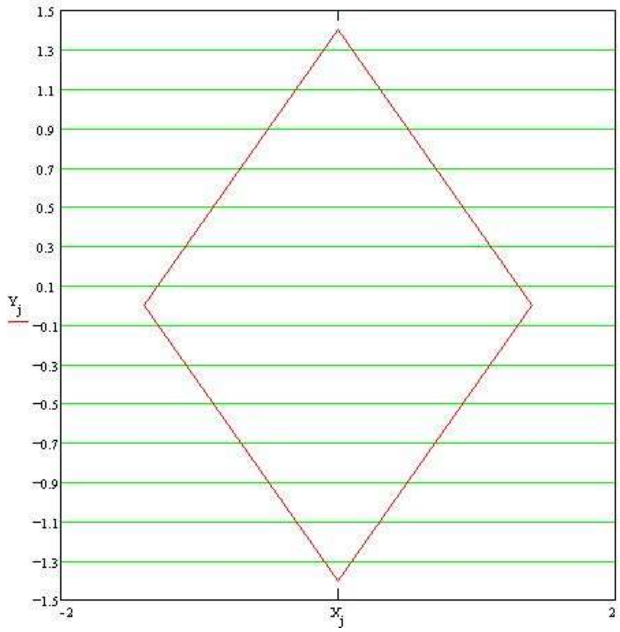

Figure 2: Periodic motion in the case $\omega_2 = \frac{4}{3}\omega_1$. It is represented in the plane X-Y the associated map, i.e. the values of the approximate solution for time t=$\frac{2\pi}{\omega}n$, where *n* is a positive integer.

3. The Approximate Second Order Solution for the Forced System

We assume that the magnitude of the external force and the detuning parameter σ is of $O(e^2)$, with

$$\omega = \Omega + \varepsilon^2\sigma \tag{46}$$

Also in this case the solution *X(t)* of equation (10) can be expressed by means of a power series in the expansion parameter as

$$X(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\gamma_n}\psi_n(\tau,|\varepsilon)\exp(-\mathrm{in}\Omega t), \tag{47}$$

where $g_n = |n|$-1for n¹0, $\gamma_0 = 1$.

Following the same steps of Sect. 2, we obtain

$$-i\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = \mathrm{N}|\psi|^2\psi + \frac{F}{2\Omega} - \sigma\psi \tag{48}$$

where N is given by (24).

Via the transformation (22), we arrive at the model equations

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \frac{F}{2\Omega}\sin\alpha, \tag{49a}$$

$$\rho\frac{\mathrm{d}\alpha}{\mathrm{d}\tau} = \mathrm{N}\rho^3 + \frac{F}{2\Omega}\cos\alpha - \sigma\rho. \tag{49b}$$

The equations (40) admit the following integral of motion

$$H(\rho,\alpha) = \rho\left(-\frac{\mathrm{N}\rho^3}{2} + \sigma\rho - \frac{F}{\Omega}\cos\alpha\right). \tag{50}$$

As a consequence the energy-like function $H(\rho,J)$ is constant along the solution curves. In order to determine the equilibrium points, we observe that the modulation equations (40) have only two independent parameters, because trough the rescaling

$$\rho \to L\rho,\ t \to Tt,\ F \to \frac{LF}{T},\ L=\sqrt{\frac{|\sigma|}{|N|}},\ T=\frac{1}{|\sigma|}, \tag{51}$$

we can always set N= ± 1, σ= ± 1. We distinguish four cases:

i) $N=1,\sigma=1$: there is only an elliptic equilibrium point, given by (Figure 3a)

$$(\rho_E,\alpha_E) = \left(\sqrt[3]{\frac{F}{2\Omega}+\sqrt{\frac{F^2}{4\Omega^2}+\frac{1}{27}}}+\sqrt[3]{\frac{F}{2\Omega}-\sqrt{\frac{F^2}{4\Omega^1}+\frac{1}{27}}},\pi\right) \tag{52}$$

ii) $N=1,\sigma=-1$: if F> $\frac{2\Omega}{3\sqrt{3}}$ there is only an elliptic equilibrium point, given by (Figure 3b)

$$(\rho_E,\alpha_E) = \left(\sqrt[3]{\frac{F}{2\Omega}+\sqrt{\frac{F^2}{4\Omega^2}-\frac{1}{27}}}+\sqrt[3]{\frac{F}{2\Omega}-\sqrt{\frac{F^2}{4\Omega^2}-\frac{1}{27}}},\pi\right) \tag{53}$$

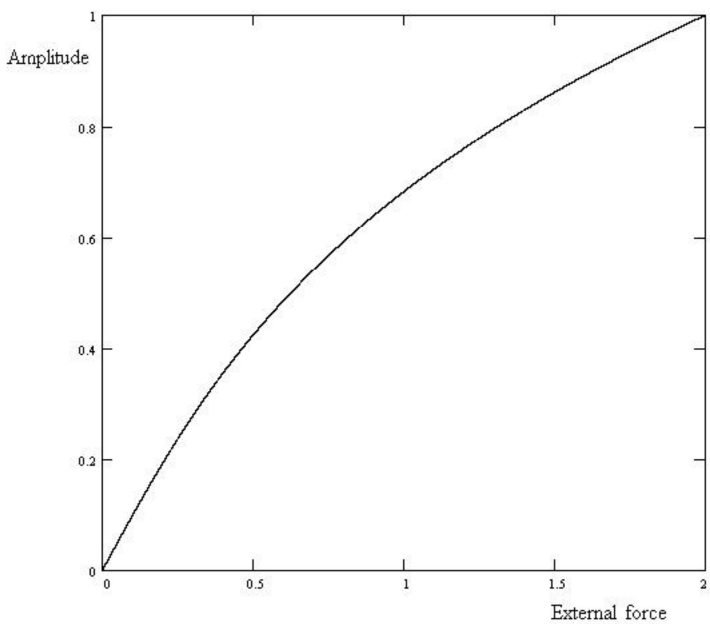

Figure 3a: Amplitude of the equilibrium point vs. the external force (case i)): the continuous line represents a stable solution.

if F< $\frac{2}{3\sqrt{3}}$, then we obtain three equilibrium points, given by (Figure 3b)

$$(\rho_{1E},\alpha_{1E}) = \left(2\sqrt[3]{R}\cos\left(\frac{\pi-\phi}{3}\right),\pi\right) \tag{54}$$

$$(\rho_{2E},J_{2E}) = \left(2\sqrt[3]{R}\cos\left(\frac{j}{3}\right),0\right), \tag{55}$$

$$\left(\rho_{3E},J_{3E}\right)=\left(2\sqrt[3]{R}\cos\left(\frac{j}{3}+\frac{4}{3}p\right),0\right), \tag{56}$$

where

$$\cos j=-\frac{3\sqrt{3}F}{2},R=\frac{1}{3\sqrt{3}}. \tag{57}$$

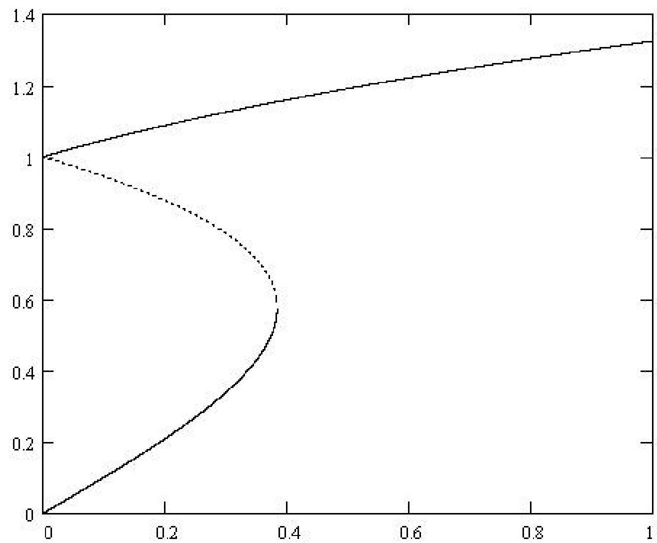

Figure 3b: Amplitudes of the equilibrium points vs. the external force (case ii)):dot line = unstable solution, continuous line = stable solution.

The equilibrium point given by equation (55) is a saddle point, while the other two equilibrium points given by equations (54) and (56) are elliptic points.

In Figure 3b, we observe an interesting behavior, because if we consider the smaller elliptic point and increase the amplitude of the external force, suddenly in correspondence with the critical value F=$\frac{2}{3\sqrt{3}}$, the solution jumps into the second elliptic point.

In Figures 4a-4b we show different solution curves in the plane (ρ,α): along each solution curve the energy-like function H(ρ,J) of the Equation (50) is constant. The case F<$\frac{2}{3\sqrt{3}}$ is represented in Figure 4a and the case F>$\frac{2}{3\sqrt{3}}$ in Figure 4b.

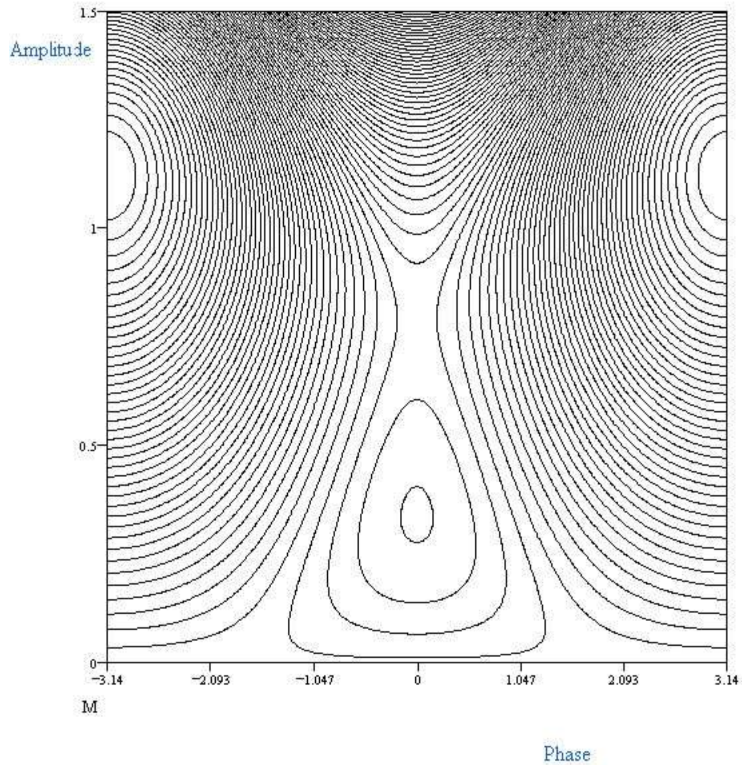

Figure 4a: Phase space representation of solutions for the system of equations (49a-b) in the case ii), F< $2/3\sqrt{3}$. Note the presence of two elliptic points and one saddle point.

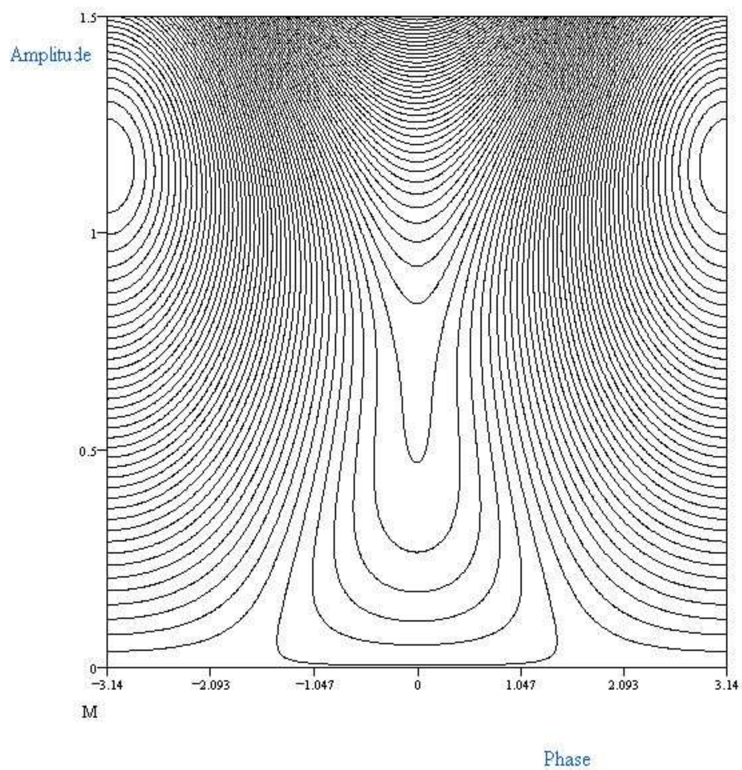

Figure 4b: Phase space representation of solutions for the system of equations (49a-b) in the case ii), F> $2/3\sqrt{3}$. The smaller elliptic point and the saddle point have disappeared.

iii) *N=-1,* σ*=-1*: there is only an elliptic equilibrium point, given by Equation (52) but with $J_E = 0$.
iv) *N=-1,* σ*=1*: if F>$\frac{2\Omega}{3\sqrt{3}}$ there is only an elliptic point, given by Equation (53) but with $J_E = 0$; if F<$\frac{2}{3\sqrt{3}}$ there are three equilibrium points, the first (saddle) and the second (elliptic) given by Equations (55) and (56) but with α_E=π, and the third, which is always elliptic, given by Equation (54) but with $\alpha_E = 0$. The last two situations are connected by a simple temporal inversion (and the transformation $\alpha \rightarrow$ π+α) to the first two cases.

The frequency of the small oscillations around the elliptic point in the plane (ρ, J) is

$$\omega_3 = \sqrt{\frac{F|3\mathrm{N}\rho^{2E} - \sigma|}{2\Omega\rho_E}}. \tag{58}$$

We observe a frequency splitting with two new frequencies Ω and ω_3 compared with the unforced case and the periodic motion around the elliptic point for the Equations (21) corresponds to a three period quasi-periodic motion for the starting Equation (10).

The approximate solution good to the order of e^2 is

$$R(t)=\psi_0 + \left(\psi\exp(-i\Omega t)+\psi_2\exp(-2i\Omega t)+\mathrm{c}.c.\right), \tag{59}$$

or more explicitly

$$R(t) = \frac{2b}{\Omega^2}\rho^2(t) + 2\rho(t)\cos\left(-\Omega\mathrm{t}+\alpha(t)\right) - \frac{2b}{3\Omega^2}\rho^2(t)\cos\left(-2\Omega\mathrm{t}+2\alpha(t)\right), \tag{60}$$

If we perform a linearization of Equations (49) near the elliptic point, we can write explicitly the solution for the small oscillations

$$\rho(t)=\rho_E + (\rho_I - \rho_E)\cos(\omega_3 t) + \frac{F\cos\alpha_E(\alpha_I-\alpha_E)}{2\omega_3\Omega}\sin(\omega_3 t), \tag{61}$$

$$\alpha(t)=\alpha_E + (\alpha_I - \alpha_E)\cos(\Omega t) + \frac{(6\mathrm{N}\rho^{2E}-\sigma)(\rho_I-\rho_E)}{2\Omega\rho_E}\sin(\Omega t), \tag{62}$$

where ρ_I,α_I are the initial conditions.

The solution of the forced system (3) and (10) is then a three period quasi-periodic motion characterized by three frequencies ω_0, Ω and ω_3. Only if ρ_I=ρ_E,α_I=α_E, the solution is simply periodic and corresponds to the elliptic equilibrium point.

Another interesting possibility which can be easily treated is the case

$$\mathrm{N}\rho^{3I} - \sigma\rho_I >> \frac{F}{2\Omega}, \tag{63}$$

because we find that the Equations (21) yield

$$\alpha(t)=\alpha_I+\omega_4 t, \tag{64}$$

$$\rho(t)=\rho_I+\mathrm{F}\left(\cos J_I - \cos(\mathrm{t}+\mathrm{J}_I)\right)\Omega 2\Omega, \tag{65}$$

where

$$\omega_4=\mathrm{N}\rho_{I^2} - \sigma. \tag{66}$$

The solution is a three period quasi-periodic motion with the frequencies ω_0, Ω and ω_4 and is always given by the equation (60).

Another particular case is the perfect resonance with the external excitation (σ=0). The rescaling (42) cannot be introduced and we simply consider the equations (40) with $\sigma=0$. We find only an elliptic equilibrium point, given by

$$\rho_E = \sqrt[3]{\frac{F}{|\alpha|}},\ \alpha_E = 0, \tag{67}$$

while another frequency splitting. The frequency of the small oscillations is

$$\omega_5 = \frac{1}{2}\sqrt{6|N|\rho_E F}. \tag{68}$$

Note that it coincides with the case $\sigma=0$ of the Equation (49). The approximate analytic solution (52-53) for the initial conditions is also valid, after taking $\sigma=0$.

4. Conclusion

The behavior of a mass point moving around the equilibrium point under the effect of a central field and an external periodic excitation in resonance with the natural frequency is studied. The asymptotic perturbation method based on temporal rescaling and balancing of the harmonic terms with a simple iteration is used in order to determine the nonlinear modulation equations for the amplitude and the phase of the oscillation. We calculate the second order approximate solution of the unforced system and demonstrate that the motion is periodic, if some Diophantine equations are satisfied. Subsequently, the forced system is considered and external force-response curves are shown and moreover jump phenomena are also observed. In two different cases a frequency splitting appears and a new frequency is present in addition to the forcing frequency and then stable three period quasi-periodic motions are present with amplitudes depending on the initial conditions. Only in the first case the low frequency depends on the amplitude of the external excitation.

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