General Relativity and the Standard Model

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Abstract
The quantum conditions are applied to yield a form of the covariant momentum. This momentum is contrasted with the covariant velocity in classical gravitational theory. The quantum momentum is then renormalized. This allows separation of mass and velocity, which velocity is considered equivalent to the classical velocity. The renormalization thereby yields two forms for the gravitational metric that are related by a gauge transformation of the first kind. The gauge entities in the metric are shown to be directly applicable to gauge generators in the Standard Model.

Keywords: General Relativity; Gauge Theory, Standard Model, Unified Field

INTRODUCTION

This paper presents a framework for introducing the gauge fields of the Standard Model into the description of the gravitational field provided by the General Theory of Relativity. This is accomplished by deriving the metric tensor of spacetime through constraints. These specify the spacetime as a surface in a system of rectilinear coordinates greater in number than the number of spacetime coordinates. Quantum conditions apply to the rectilinear coordinates. To be meaningful, the quantum conditions require quantum states of momentum. Separating mass from momentum, thereby describing velocity of particles with mass, requires renormalization of the quantum momentum states. This is done by restricting the range of integration of quantum densities. As a result, the scale of the quantum states is changed by a multiplicative factor. If the factor is the same for all momentum components, the equations of motion for the resulting massive particle include both the gauge field and the gravitational contributions. The way the gauge field generators of the Standard Model are related to the gauge forms introduced into general relativity is shown explicitly.

I. QUANTUM CONDITIONS

Let y^I denote the coordinates of the rectilinear space with more dimensions than spacetime, and x^I the coordinates in this space that coincide with spacetime. Let x^μ $(\mu = 0,1,2,3)$ denote the curvilinear spacetime coordinates. The covariant metric tensor is

$$g_{\mu\nu} = x^I{}_{,\mu} x^J{}_{,\nu} g_{IJ}, \tag{1}$$

where the g_{IJ} are constants, the subscript commas denote ordinary differentiation, and the $x^I{}_{,\mu}$ are transformation laws that embody the equations of constraint that specify the spacetime coordinates [1]. In four-dimensional (4-d) spacetime, the metric has a signature, $(+,-,-,-)$, appropriate for one time (+) coordinate and three spatial (-) coordinates.

The coordinate x^I is considered to be conjugate to a quantum momentum operator $\hat{p}_I$,

$$\hat{p}_I = -i\hbar\hat{\partial}_I,$$

where $i = \sqrt{-1}$, $\hbar$ is the reduced Planck's constant $\frac{h}{2\pi}$, and $\hat{\partial}_I$ operates to take the partial derivative with respect to x^I. The momentum operator gives the quantum conditions by its commutation properties with the coordinates,

$$[x^I, \hat{p}_J] = x^I\hat{p}_J - (\hat{p}_J x^I + x^I\hat{p}_J) = i\hbar\delta^I_J,$$

where δ^I_J is the Kroenecker delta. The eigenfunction of the quantum momentum operator is Ψ, with eigenvalues p_I. These eigenvalues are constants.

The eigenfunction is normalized to unity over the entire 4-d spacetime volume Ω of differential volume element dV,

$$\int_0^\Omega dV\Psi^*\Psi = 1,$$

where Ψ^* is the complex conjugate of Ψ. In cases beyond the scope of the present paper, Ψ becomes a multi-component wavefunction, written as a vector. So Ψ^* is replaced therein by $\Psi^\dagger$, the complex transpose of the wavefunction.

The volume of spacetime is not the entire volume of the rectilinear space given by the y^I. Rather, it is the volume in the full rectilinear space formed by the x^I. This volume is a functional of the spacetime coordinates x^μ by virtue of the equations of constraint. Set this functional of the x^μ to be $\Omega[f(x^\mu)]$, where $f(x^\mu)$ represents the equations of constraint. A suitable form for the eigenfunction is

$$\Psi(x^I) = \Omega^{-\frac{1}{2}} exp\left[\frac{ix^I p_I}{\hbar}\right]. \tag{2}$$

The corresponding spacetime momenta are $p_\mu = x^I{}_{,\mu} p_I$. As the spacetime coordinates are curvilinear, the p_μ can change with the coordinates because the $x^I{}_{,\mu}$ can change with the coordinates. In the quantum case, the operator is transformed rather than the eigenvalues. This allows the operator $\hat{p}_\mu = x^I{}_{,\mu} \hat{p}_I$ to act on the wavefunction (2). The expectation value of the momentum is

$$p_\mu = \int_0^\Omega dV \Psi^\dagger x^I{}_{,\mu} \hat{p}_I \Psi = p_I \int_0^\Omega dV \Psi^\dagger x^I{}_{,\mu} \Psi. \tag{3}$$

The quantity $\Psi^\dagger x^I{}_{,\mu} \Psi$ is a density. As $x^I{}_{,\mu}$ takes no derivative in the integrand, Eq. (3) becomes

$$p_\mu = p_I \int_0^\Omega dV \Omega^{-1} x^I{}_{,\mu}.$$

The integrand itself is a function only of the x^I. By the fundamental theorem of integral calculus, the x^μ-dependence enters through the upper limit, $\Omega[f(x^\mu)]$, provided only that $x^I \Omega^{-1}$ is a total derivative of some function $N^I(x^I)$ with V. Then

$$\int_0^\Omega dV \Omega^{-1} x^I = \int_0^\Omega dV \frac{dN^I}{dV} = N^I\{\Omega[f(x^\mu)]\} - N^I(0)$$

$$= N^I(x^\mu) - const, \tag{4}$$

where $N^I(0)$ is constant. This allows the transformation law to be a function only of the x^μ, and to be consistent with the equations of constraint $f(x^\mu)$. Specifically,

$$x^I{}_{,\mu}(x^\nu) = [N^I(x^\nu)]_{,\mu}. \tag{5}$$

Then Eq. (3) yields the spacetime momentum expectation values

$$p_\mu(x^\nu) = p_I [N^I(x^\nu)]_{,\mu} = p_I x^I{}_{,\mu}(x^\nu).$$

II. CLASSICAL VELOCITIES

With the metric tensor, the classical gravitational equations of motion for a massive particle are

$$\frac{dv_\mu}{ds} = \Gamma^\tau_{\mu\nu} v_\tau v^\nu = \Gamma_{\lambda\mu\nu} v^\lambda v^\nu$$

along a timelike geodesic path of squared differential interval $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, and for which the velocities are $v_\mu = g_{\mu\nu} \frac{dx^\nu}{ds} = g_{\mu\nu} v^\nu$. The geodesic equations of motion are consistent with the principal of least action, and with the vanishing of the covariant total derivative of velocity,

$$\frac{Dv_\mu}{ds} = v_{\mu:\nu} v^\nu = \left(v_{\mu,\nu} - \Gamma^\lambda_{\mu\nu} v_\lambda\right) v^\nu = \frac{dv_\mu}{ds} - \Gamma_{\lambda\mu\nu} v^\lambda v^\nu = 0.$$

Here the subscript colon denotes covariant differentiation and the $\Gamma_{\lambda\mu\nu}$ are the Christoffel symbols of the first kind,

$$\Gamma_{\lambda\mu\nu} = \left(\frac{1}{2}\right)\left(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\right). \tag{6}$$

Note that using the covariant derivative of the contravariant velocity, $v^{\mu}{}_{:\nu} = v^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\alpha\nu} v^{\alpha}$, gives

$$g_{\mu\lambda}\frac{dv^{\lambda}}{ds} = -g_{\mu\lambda}\Gamma^{\lambda}_{\alpha\nu} v^{\alpha} v^{\nu} = -\Gamma_{\mu\alpha\nu} v^{\alpha} v^{\nu},$$

where the $\Gamma^{\tau}_{\mu\nu} = g^{\tau\lambda}\Gamma_{\lambda\mu\nu}$ are Christoffel symbols of the second kind, or metric connections. Setting $v_{\mu:\nu}v^{\nu} = g_{\mu\lambda}v^{\lambda}{}_{:\nu}v^{\nu}$ gives two expressions for the equations of motion.

As characteristic of a gravitational field description, the classical geodesic equations of motion do not involve mass explicitly. The quantum momentum p_{μ} does not contain a definition of velocity v_{μ}. This is because mass is not separable from momentum using the quantum conditions alone. In order to compare the quantum momentum to the classical velocity, a mass m is required. As the classical velocity v_{μ} is dimensionless, the speed of light c is needed to give traditional momentum units. Then a classical momentum defined as mcv_{μ} can be compared to p_{μ}.

III. RENORMALIZATION

In order to establish mass with a timelike interval, the spacetime field is renormalized. This is done by evaluating the integral in Eq. (3) between limits L_1 and L_2, where $0 \leq L_1 \leq L_2 \ll \Omega$, instead of the total spacetime limits. The integrated momentum density contained within this renormalized volume is a fraction, η, of the total momentum (3). Renormalization consists of converting the wavefunction into one for which the renormalized eigenvalue is the eigenvalue of the momentum operator, and which is normalized over the entire spacetime. The components of such a wavefunction Φ are

$$\Phi = \Omega^{-\frac{1}{2}} exp\left[\frac{ix^{I}\eta p_{I}}{\hbar}\right].$$

So Eq. (3) is recast as a renormalized momentum $\pi_{\mu'}$

$$\pi_{\mu'} = \delta^{\mu}_{\mu'} p_{I}\int_{L_1}^{L_2} dV \Psi^{\dagger} x^{I}{}_{,\mu}\Psi = \eta p_{I}\int_{0}^{\Omega} dV \Psi^{\dagger} x^{I}{}_{,\mu}\delta^{\mu}_{\mu'}\Psi$$

$$= \int_{0}^{\Omega} dV \Phi^{\dagger} x^{I}{}_{,\mu}\delta^{\mu}_{\mu'}\hat{p}_{I}\Phi = \eta^{\mu}_{\mu'} p_{\mu}, \tag{7}$$

where $\eta^{\mu}_{\mu'} = \eta\delta^{\mu}_{\mu'}$.

With reference to Eqs. (4) and (5), condition (7) requires a constraint on L_2 as a function of the x^{μ} such that

$$(N^{I}\{L_2[f'(x^{\mu})]\})_{,\mu} = \eta(N^{I}\{\Omega[f(x^{\mu})]\})_{,\mu}. \tag{8}$$

Because η is a scalar factor, the components of p_μ are all rescaled together. The limit as L_1 and L_2 approach each other, $\lim_{L_2 \to L_1} p_I \int_{L_1}^{L_2} dV\, \Psi^\dagger x^I{}_{,\mu} \Psi$, might be considered as a point particle limit. However, physical constraints may dictate how close this limit can be approached. That is, constraints may demand a minimum allowed difference $L_2 - L_1$.

The renormalized metric tensor is $g_{\mu'\nu'} = \eta^\mu_{\mu'} \eta^\nu_{\nu'} g_{\mu\nu}$. To yield an invariant squared differential interval, the renormalized classical coordinate differential must be $dx^{\mu'} = \eta^{-1} \delta^{\mu'}_\mu dx^\mu$.

Renormalization allows for separating a mass m from the quantum momentum as a scalar factor. With this separable mass, $p_I = mcv_I$ at points in the region of renormalization, and the equations of motion are expressible using the quantum quantities $\frac{p_\mu}{mc}$, not just the classical velocities $g_{\mu\nu} \frac{dx^\nu}{ds}$.

The explicit way that renomalization allows separating mass is based on the constant values of the eigenvalues p_I. If the scale factor is constant, then ηp_I is a product of two constants. The constant mc is introduced as a constant with momentum units. This constant is incorporated into a new constant scale factor P and a dimensionless velocity v_I,

$$\eta p_I = P \left(\frac{p_I}{mc} \right) = (mc\eta) \left(\frac{p_I}{mc} \right) = P v_I.$$

In this way, $\frac{P}{c} = m\eta$ has units of mass, is constant, and is separated from a quantum velocity $v_I = \frac{p_I}{mc}$. Because the quantum velocity does not depend on the scale factor P, its behavior can be analyzed without reference to the renormalization scale. Accordingly, the scale factor η and the reference to the corresponding transformation $\eta^\eta_{\mu'}$ is dropped and the mass m retained. At this point in the development of the theory, the behavior of the renormalized quantum velocities $x^I{}_{,\mu} v_I$ is equivalent to the behavior of the coordinate velocities $\frac{dx_\mu}{ds}$ within the general theory of relativity.

IV. GAUGE FROM RENORMALIZATION

Given Eq. (8), dropping the reference to the equations of constraint, define coordinate-like quantities $M^I(\Omega, L_2)$ as

$$M^I(\Omega, L_2) = N^I(\Omega) - N^I(L_2) = (1 - \eta) N^I(\Omega).$$

Taking the derivatives with the x^μ, multiplying by η and rearranging gives the transformation law $x^I{}_{,\mu} = \eta^{-1} \left(x^I{}_{,\mu} - M^I{}_{,\mu} \right)$. The spacetime momentum is

$$p_\mu = p_I x^I{}_{,\mu} = \eta^{-1} \left(p_\mu - r_\mu \right),$$

where $r_\mu = p_I M^I{}_{,\mu}$. Now Eq. (7) gives a renormalized momentum $\pi_{\mu'}$,

$$\pi_{\mu'} = \eta^\mu_{\mu'} p_\mu = \delta^\mu_{\mu'} \left(p_\mu - r_\mu \right) = p_{\mu'} - r_{\mu'},$$

in which the scale factor does not appear. Because the transformation $\pi_{\mu'} = \delta^{\mu}_{\mu'}\pi_{\mu}$ holds for the renormalized momenta, the reference to the $x^{\mu'}$ can be dropped using the renormalized momenta.

The renormalized metric tensor is

$$g_{\mu'\nu'} = g_{IJ}x^{I}{}_{,\mu}x^{J}{}_{,\nu}\eta^{\mu}_{\mu'}\eta^{\nu}_{\nu'} = g_{IJ}\eta^{-2}\left(x^{I}{}_{,\mu} - M^{I}{}_{,\mu}\right)\left(x^{J}{}_{,\nu} - M^{J}{}_{,\nu}\right)\eta^{\mu}_{\mu'}\eta^{\nu}_{\nu'}$$

$$= g_{IJ}\left(x^{I}{}_{,\mu} - M^{I}{}_{,\mu}\right)\left(x^{J}{}_{,\nu} - M^{J}{}_{,\nu}\right)\delta^{\mu}_{\mu'}\delta^{\nu}_{\nu'} = g_{IJ}\left(x^{I}{}_{,\mu'} - M^{I}{}_{,\mu'}\right)\left(x^{J}{}_{,\nu'} - M^{J}{}_{,\nu'}\right). \quad (9)$$

The scale factor does not appear in the bottom line of Eq. (9). Consequently, reference to the $x^{\mu\prime}$ can be dropped using the renormalized metric tensor. The renormalization transformation law is $\eta x^{I}{}_{,\mu} = x^{I}{}_{,\mu} - M^{I}{}_{,\mu}$, and the renormalized metric tensor is $G_{\mu\nu} = \eta^2 g_{\mu\nu}$. The renormalized squared differential interval is formed with the dx^{μ}, giving

$$\eta^2 ds^2 = g_{\mu'\nu'}\delta^{\mu'}_{\mu}\delta^{\nu'}_{\nu}dx^{\mu}dx^{\nu} = G_{\mu\nu}dx^{\mu}dx^{\nu}$$

$$= g_{IJ}(dx^I - dM^I)(dx^I - dM^I).$$

After renormalization, the transformation laws between coordinates remain $x^{I}{}_{,\mu}$ such that

$$g_{\mu\nu} = g_{IJ}x^{I}_{,\mu}x^{J}_{,\nu}$$

and

$$dx^{\mu} - dM^{\mu} = g^{\mu\alpha}x_{J,\alpha}(dx^J - dM^J),$$

giving the renormalized squared differential interval as

$$\eta^2 ds^2 = g_{\mu\nu}(dx^{\mu} - dM^{\mu})(dx^{\nu} - dM^{\nu}) = G_{\mu\nu}dx^{\mu}dx^{\nu}. \quad (10)$$

Because the dM^I and $M^I_{,\mu}$represent a scale factor, these quantities represent a gauge field. The two forms for ds^2 in Eq. (10) are related by a gauge transformation of the first kind, wherein the $M^I_{,\mu}$ are expressed in the renormalized metric tensor $G_{\mu\nu}$ in one form, while they enter though the renormalized coordinate differentials in the other gauge form.

The principle of least action is satisfied by either gauge form consistently through the variation of the differential interval,

$$\delta\int_{l_1}^{l_2} ds = \frac{1}{\eta}\int_{l_1}^{l_2} ds\,\frac{\delta G_{\mu\nu}dx^{\mu}dx^{\nu}}{2ds^2} = 0. \quad (11)$$

As the scale factor is not varied, the renormalized squared differential interval is varied with the variations δx^{μ} of the coordinates. The variations δx^{μ} vanish at limits l_1 and l_2. In the explicit gauge form in Eq. (11), the resulting equations of motion are

$$g_{\mu\lambda}\frac{dv^{\lambda}}{ds} = -\frac{1}{2}\left(G_{\nu\mu,\lambda} + G_{\mu\lambda,\nu} - G_{\mu\lambda,\nu}\right)v^{\lambda}v^{\nu} = -T_{\mu\nu\lambda}v^{\lambda}v^{\nu}. \quad (12)$$

Here $T_{\mu\nu\lambda}$ has the form of a Christoffel symbol of the first kind written in terms of the renormalized metric tensor $G_{\mu\nu}$. In the other gauge form, the partial derivatives are taken with respect to the $x^{\nu} - M^{\nu} = \eta x^{\nu}$, and the variations $\delta(x^{\nu} - M^{\nu})$ are used along with the ordinary metric tensor $g_{\mu\nu}$. Define the gauge velocity as $V^{\mu} = \frac{d(x^{\nu}-M^{\nu})}{ds} = v^{\nu} - w^{\nu}$, where w^{ν} is the velocity associated with the coordinate-like M^{ν}. The variation is carried out using the following additional relationships:

$$\frac{\partial g_{\mu\lambda}}{\partial(x^{\nu}-M^{\nu})} = \eta^{-1} g_{\mu\lambda,\nu},$$

$$\delta g_{\mu\nu} = g_{\mu\nu,\lambda}\delta x^{\lambda} = \left[\frac{\partial g_{\mu\nu}}{\partial(x^{\lambda}-M^{\lambda})}\right]\delta\left(x^{\lambda} - M^{\lambda}\right) = \eta^{-1} g_{\mu\nu,\lambda}\delta\left(x^{\lambda} - M^{\lambda}\right),$$

and

$$\left[\frac{\partial g_{\mu\nu}}{\partial(x^{\lambda}-M^{\lambda})}\right] V^{\lambda} = g_{\mu\nu,\lambda} v^{\lambda} = \eta^{-1} g_{\mu\nu,\lambda} V^{\lambda}.$$

The result is

$$g_{\mu\lambda}\frac{dv^{\lambda}}{ds} = g_{\mu\lambda}\frac{dw^{\lambda}}{ds} - \Theta_{\mu\nu\lambda} V^{\lambda} V^{\nu},$$

where $\Theta_{\nu\mu\lambda}$ is the gauge form of the Christoffel symbol of the first kind written with the $\eta^{-1} g_{\mu\lambda,\nu}$.

V. QUANTUM GAUGE FIELDS

The preceding demonstrated the way in which the quantum mechanical property of momentum is converted into the classical property of velocity and incorporated into general relativity. This introduced a scale factor that led to the quantities dM^{I} and $M^{I}_{,\mu}$that represent a gauge field.

The Standard Model of particle physics is a quantum theory based on the use of generators of gauge transformations of the momentum. The connection between the scale factors introduced in the preceding and the quantum mechanical gauge generators is presented in the following.

The renormalized wavefunction Φ contains the scale factor η. This renormalized wavefunction is expressed as a product of the original wavefunction, Ψ, and a gauge-generating function, ϕ, such that

$$\Phi = \Omega^{-\frac{1}{2}} exp\left(\frac{i\eta x^{I} p_{I}}{\hbar}\right) = \Omega^{-\frac{1}{2}} exp\left(\frac{i x^{I} p_{I}}{\hbar}\right) exp\left[-i(1-\eta)\frac{x^{I} p_{I}}{\hbar}\right] = \Psi\phi.$$

Here Ψ is given by Eq. (2) and the gauge-generating function is

$$\phi = exp\left[-i(1-\eta)\frac{x^{I} p_{I}}{\hbar}\right].$$

With a gauge generator θ given by $\theta = (1-\eta)\frac{x^{I} p_{I}}{\hbar}$, the gauge-generating function takes a convenient form, $\phi = exp(-i\theta)$. This generates a momentum that is consistent with that of Eq. (7),

$$\pi_\mu = \int_0^\Omega dV \Phi^\dagger x^I{}_{,\mu} p_I \Phi = x^I{}_{,\mu} p_I \left[1 + (\eta - 1)\right] = x^I{}_{,\mu}\left(p_I - \hbar\theta,_I\right) = p_\mu - r_\mu .$$

This preserves the interpretation of the scale factor-related quantities $M^I_{,\mu}$ such that $r_\mu = p_I M^I_{,\mu} = \hbar\theta,_\mu$ is the momentum associated with the gauge field.

The above introduction of the gauge generator leads to the gauge-generalized momentum operator,

$$\hat{\pi}_\mu = x^I{}_{,\mu}\left(\hat{p}_I - \hbar\hat{\theta},_I\right),$$

that acts on the wavefunction Ψ. Gauge generators and gauge-generalized momentum operators are tools upon which the Standard Model is built.

References

[1] Paul Adrien Maurice Dirac, *General Theory of Relativity* (Wiley, New York, 1975).