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Abundant Exact Traveling Wave Solutions of the (2+1)-Dimensional Couple Broer-Kaup Equations

Mohammad Mobarak Hossain[1,3,*], Harun-Or-Roshid[2] and Md. Abu Naim Sheikh[3]

[1]Department of Mathematics, Hamdard University Bangladesh, Bangladesh
[2]Department of Mathematics, Pabna University of Science and Technology, Bangladesh
[3]Department of Mathematics, Dhaka University of Engineering & Technology, Gazipur, Bangladesh

**Corresponding author E-mail*: mobarak4074@gmail.com

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Abstract

To describe the propagation of small amplitude waves in nonlinear dispersive media, it is frequently necessary to take account of dissipative mechanisms to perfectly reflect real situations in many branches of physics like plasma physics, fluid dynamics and nonlinear optics. In this paper, the $\exp(-\phi(\eta))$-expansion method is employed to solve the (2+1)-Dimensional couple Broer-Kaup equations as a model for wave propagation in nonlinear media with dispersive and dissipative effects. As a result, a number of exact traveling wave solutions including solitary wave and periodic wave have been found for the equation. Some representative 3D profiles and 2D profiles for different values of variables of the wave solutions are graphically displayed and discussed.

***Keywords*:** The $\exp(-\phi(\eta))$-expansion method; traveling wave solutions; the (2+1)-dimensional couple Broer-Kaup equations; nonlinear evolution equations.

1. Introduction

The world around us is essentially nonlinear and nonlinear partial differential equations (NPDE) are broadly used as models to illuminate the complex physical phenomena. The exact traveling wave solutions of NPDEs play a vital role in nonlinear science and engineering. The nonlinear evolution equations (NLEF) are involved in many fields such as mathematics, physics, biology, chemistry, mechanics, meteorology, engineering, optical fibers etc. The investigation of the exact traveling wave solutions of nonlinear evolutions equations play an important role in the study of nonlinear physical phenomena. In the recent decade, many methods were developed and proposed for finding the exact solutions of nonlinear evolution equations, such as the modified extended Fan sub-equation method [1], the homogeneous balance method [2-3], the tanh method [4-5], the Jacobi elliptic function expansion [6], the F-expansion method [7,8], the Backlund transformation method [9], the Darboux transformation method [10], the Adomian decomposition method [11-13], the auxiliary equation method [14, 15] and the (G'/G)-expansion method [16-23]. Recently, authors in [24, 25] have obtained the exact traveling wave solutions of some Nonlinear Evolution Equations using the $\exp(-\phi(\eta))$-expansion method. It will be seen that more traveling wave solutions of many nonlinear evolution equations can be obtained by using the $\exp(-\phi(\eta))$-expansion method.

In the present article, we apply the $\exp(-\phi(\eta))$-expansion method to find some exact new traveling wave solutions of the (2+1)-Dimensional Broer-Kaup equations.

2. Methodology

In this section, we describe $\exp(-\phi(\eta))$- expansion method for finding traveling wave solutions of nonlinear evolution equations. Suppose that a nonlinear equation, say in three independent variables x, y and t is given by

$$\left.\begin{aligned} F(H,G,H_t,G_t,H_x,G_x,H_y,G_y,H_{tt},G_{tt},H_{xx},G_{xx},H_{xt},..................)=0 \\ T(H,G,H_t,G_t,H_x,G_x,H_y,G_y,H_{tt},G_{tt},H_{xx},G_{xx},H_{xt},..................)=0 \end{aligned}\right\} \quad (1)$$

where $u(\eta)=u(x,y,t)$ is an unknown function, F is a polynomial of $u(x,y,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method:

Step 1: Combining the independent variables x, y and t into one variable $\eta = x + y \pm wt$ we suppose that,

$$H(x,y,t)=H(\eta), \quad G(x,y,t)=G(\eta), \eta = x+y \pm wt \quad (2)$$

The travelling wave transformation Eq. (2) permits us to reduce Eq. (1) to the following ordinary differential equation (ODE):

$$\left.\begin{aligned} \Re(H,G,H',G',H'',G''.........)=0 \\ \aleph(H,G,H',G',H'',G''.........)=0 \end{aligned}\right\} \quad (3)$$

where $\Re$, $\aleph$ is a polynomial in $H(\eta), G(\eta)$ and its derivatives, whereas $H'(\eta)=\frac{dH}{d\eta}$, $H''(\eta)=\frac{d^2H}{d\eta^2}$ and so on.

Step 2: We suppose that Eq.(3) has the formal solution

$$\left.\begin{aligned} H(\eta) = \sum_{i=0}^{n} A_i (\exp(-\phi(\eta)))^i \\ G(\eta) = \sum_{i=0}^{m} B_i (\exp(-\phi(\eta)))^i \end{aligned}\right\} \quad (4)$$

Where $A_i (0 \le i \le n)$ and $B_i (0 \le i \le m)$ are constants to be determined, such that $A_n \ne 0$ and $B_m \ne 0$ and $\phi = \phi(\eta)$ satisfies the following ODE:

$$\phi'(\eta) = \exp(-\phi(\eta)) + \mu \exp(\phi(\eta)) + \lambda \quad (5)$$

Eq. (5) gives the following solutions:

When $\lambda^2 - 4\mu > 0, \mu \ne 0$,

$$\phi(\eta) = \ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}\right)(\eta+E) - \lambda}{2\mu}\right) \quad (6)$$

When $\lambda^2 - 4\mu < 0, \mu \neq 0$,

$$\phi(\eta) = \ln\left(\frac{\sqrt{(4\mu - \lambda^2)} \tan\left(\frac{\sqrt{(4\mu - \lambda^2)}}{2} (\eta + E) \right) - \lambda}{2\mu} \right) \tag{7}$$

When $\lambda^2 - 4\mu > 0, \mu = 0, \lambda \neq 0$,

$$\phi(\eta) = -\ln\left(\frac{\lambda}{\exp(\lambda(\eta + E)) - 1} \right) \tag{8}$$

When $\lambda^2 - 4\mu = 0, \mu \neq 0, \lambda \neq 0$,

$$\phi(\eta) = \ln\left(-\frac{2(\lambda(\eta + E) + 2)}{\lambda^2 (\eta + E) - 1} \right), \tag{9}$$

When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0$,

$$\phi(\eta) = \ln(\eta + E) \tag{10}$$

$A_n, B_m .....w, \lambda, \mu$ are constants to be determined later, $A_n \neq 0$ and $B_m \neq 0$, the positive integer n and m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (3).

Step 3: Inserting Eq. (4) into Eq. (3) and then we account the function $\exp(-\phi(\eta))$. As a result of this substitution, we get a polynomial of $\exp(-\phi(\eta))$. We equate all the coefficients of same power of $\exp(-\phi(\eta))$ to zero. This technique yields a system of algebraic equations whichever can be solved to find $A_n, B_m .....w, \lambda, \mu$. Substituting the values of $A_n, B_m .....w, \lambda, \mu$ into Eq. (4) along with general solutions of Eq. (5) determine the solution of Eq. (1).

3. Application of the method:

When attempting to describe the propagation of small amplitude waves in nonlinear dispersive media, it is frequently necessary to take account of dissipative mechanisms to perfectly reflect real situations. In this section, we study the following (2+1)-Dimensional couple Broer-Kaup equations [1] as a model for wave propagation in nonlinear media with dispersive and dissipative effects:

$$\left. \begin{aligned} H_{yt} - H_{xxy} + 2(HH_x)_y + 2G_{xx} = 0, \\ G_t + G_{xx} + 2(HG)_x = 0 \end{aligned} \right\} \tag{11}$$

where $H(x, y, t)$ and $G(x, y, t)$ are functions depending on the spatial variables x, y and temporal variable t. This system contains both linear dispersive term H_{xxy}, dissipative terms H_{yt}, G_{xx}, nonlinear dissipative term $(HH_x)_y$ and has been widely applied to many branches of physics like plasma physics, fluid dynamics, and nonlinear optics and so on. Yomba [1] has obtained new and more general solutions of (11) including a series of non-traveling wave and coefficient function solutions using the modified extended Fan sub equation method.

Let us now solve (1) by the $\exp(-\phi(\eta))$- expansion method. We utilize the traveling wave variable $H(\eta) = H(x, y, t), G(\eta) = G(x, y, t)$, $\eta = x + y - wt$, Eq. (11) is carried into following ODEs:

$$\left.\begin{aligned} &-wH''-H'''+2(HH')'+2G''=0 \\ &-wG'+G''+2(HG)'=0 \end{aligned}\right\} \quad (12)$$

Integrating (12) with respect to η once yields

$$K_1-wH'-H''+2HH'+2G'=0 \quad (13)$$

$$K_2-wG+G'+2HG=0 \quad (14)$$

where K_1 and K_2 are integration constants. Considering the homogeneous balance between highest order derivatives and nonlinear terms in Equations (13), (14) we deduce that

$$H(\eta)=A_0+A_1(\exp(-\phi(\eta))) \quad (15)$$

$$G(\eta)=B_0+B_1(\exp(-\phi(\eta)))+B_2(\exp(-\phi(\eta)))^2 \quad (16)$$

Switching Eq. (15) and Eq. (16) into Eq. (13) and then equating the coefficients of $\exp(-\Phi(\eta))$ to zero, we get

$$\left.\begin{aligned} &-2B_1\mu-A_1\mu\lambda+K_1-2A_1A_0\mu-wA_1\mu=0 \\ &-2B_1\lambda-4B_2\mu-2A_1{}^2\mu-A_1\lambda^2-2A_1\mu+wA_1\lambda-2A_0A_1\lambda=0 \\ &-2B_1-4B_2\lambda-2A_1A_0-2A_1{}^2\lambda-3A_1\lambda+wA_1=0 \\ &-4B_2-2A_1{}^2-2A_1=0 \end{aligned}\right. \quad (17)$$

Switching Eq. (15) and Eq. (16) into Eq. (14) and then equating the coefficients of $\exp(-\phi(\eta))$ to zero, we get

$$\left.\begin{aligned} &-B_1\mu+2A_0B_0+K_2-wB_0=0 \\ &-B_1\lambda+2A_1B_0+2A_0B_1-2B_2\mu-wB_1=0 \\ &-B_1+2A_1B_1+2A_0B_2-2B_2\lambda-wB_2=0 \\ &2A_1B_2-2B_2=0 \end{aligned}\right. \quad (18)$$

Solving the Eq. (17)-Eq. (18) together yields

$$w=2A_0-\lambda,\ A_0=const,\ A_1=1,\ B_0=-\mu,\ B_1=-\lambda,\ B_2=-1,\ K_1=K_2=0$$

where λ,μ are arbitrary constants.

Now substituting the values of w, A_o, A_1 into Eq. (15) yields

$$H(\eta)=A_0+(\exp(-\phi(\eta))) \quad (19)$$

Again substituting the values of w, B_o, B_1, B_2 into Eq. (16) yields

$$G(\eta)=-\mu-\lambda\times\exp(-\phi(\eta))-(\exp(-\phi(\eta)))^2 \quad (20)$$

where $\eta=x+y-(2A_0-\lambda)t$

Now, inserting Eq. (6) - Eq. (10) into Eq. (19) and Eq. (20) respectively, we get the following ten traveling wave solutions of the (2+1)-Dimensional couple Broer-Kaup equations.

When $\mu\neq 0,\ \lambda^2-4\mu>0,$

$$H_1(\eta)=A_0-\frac{2\mu}{\sqrt{(\lambda^2-4\mu)}\tanh(\frac{\sqrt{(\lambda^2-4\mu)}}{2})(\eta+E)+\lambda}$$

$$G_1(\eta) = -\mu + \frac{2\lambda\mu}{\sqrt{(\lambda^2-4\mu)}\tanh(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\eta+E))+\lambda} - \left(\frac{2\mu}{\sqrt{(\lambda^2-4\mu)}\tanh(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\eta+E))+\lambda}\right)^2$$

where $\eta = x + y - (2A_0 - \lambda)t$ and A_0, E are arbitrary constants

When $\mu \neq 0,\ \lambda^2 - 4\mu < 0,$

$$H_2(\eta) = A_0 + \frac{2\mu}{\sqrt{(4\mu-\lambda^2)}\tan(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\eta+E))-\lambda}$$

$$G_2(\eta) = -\mu - \frac{2\lambda\mu}{\sqrt{(4\mu-\lambda^2)}\tan(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\eta+E))-\lambda} - \left(\frac{2\mu}{\sqrt{(4\mu-\lambda^2)}\tan(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\eta+E))-\lambda}\right)^2$$

where $\eta = x + y - (2A_0 - \lambda)t$ and A_0, E are arbitrary constants.

When $\mu = 0,\ \lambda \neq 0,$ and $\lambda^2 - 4\mu > 0,$

$$H_3(\eta) = A_0 + \frac{\lambda}{\exp(\lambda(\eta+E))-1}$$

$$G_3(\eta) = -\frac{\lambda^2}{\exp(\lambda(\eta+E))-1} - \left(\frac{\lambda}{\exp(\lambda(\eta+E))-1}\right)^2$$

where $\eta = x + y - (2A_0 - \lambda)t$ and A_0, E are arbitrary constants.

When $\mu \neq 0,\ \lambda \neq 0,$ and $\lambda^2 - 4\mu = 0,$

$$H_4(\eta) = A_0 - \frac{\lambda^2(\eta+E)-1}{2(\lambda(\eta+E)+2)}$$

$$G_4(\eta) = -\mu + \lambda \times \frac{\lambda^2(\eta+E)-1}{2(\lambda(\eta+E)+2)} - \left(\frac{\lambda^2(\eta+E)-1}{2(\lambda(\eta+E)+2)}\right)^2$$

where $\eta = x + y - (2A_0 - \lambda)t$ and A_0, E are arbitrary constants. The figures of the solutions $H_4(\eta), G_4(\eta)$ are similar to the figure of the solution $H_3(\eta)$.

When $\mu = 0,\ \lambda = 0,$ and $\lambda^2 - 4\mu = 0,$

$$H_5(\eta) = A_0 + \frac{1}{\eta+E}$$

$$G_5(\eta) = -\left(\frac{1}{\eta+E}\right)^2$$

where $\eta = x + y - 2A_0 t$ and A_0, E are arbitrary constants.

4. Graphical representation of the solutions

We study here the interaction of two wave solutions to the (2+1)-Dimensional couple Broer-Kaup equations. The graphical illustrations of the solutions are depicted in the figures from fig-1 to fig-5 with the aid of commercial software Maple 13, where all the figures are estimated along $y = 0$ and compared with analytical solution cases for $\lambda = 5, \mu = 1.25$ such that $\lambda^2 - 4\mu > 0$ and $A_0 = 0, E = 1$. Numerical representation produces that the same behavior as wave solutions. The solutions persist before and after their interaction. As seen in Fig.1(a) of $H_1(\eta)$ is Kink waves that are traveling waves which arise from one asymptotic state to another. The kink solutions approach to a constant at infinity. In 2D profile seen in Fig. 1(b), for time evolution of $H_1(\eta)$ wave for different values of displacement on the domain [0, 1] (see t from 0 to 1 only), we see that $H_1(\eta)$ wave varies with displacement. It is found that the wave flow oscillates regularly that is periodic over the displacement region $0 \le x \le 9$. Fig. 1(c) shows the $H_1(\eta)$ wave for different values of time t for the whole region of displacement $-3 \le x \le 3$ and time $0.2 \le t \le 0.8$. It is seen that the wave increases gradually as time increases.

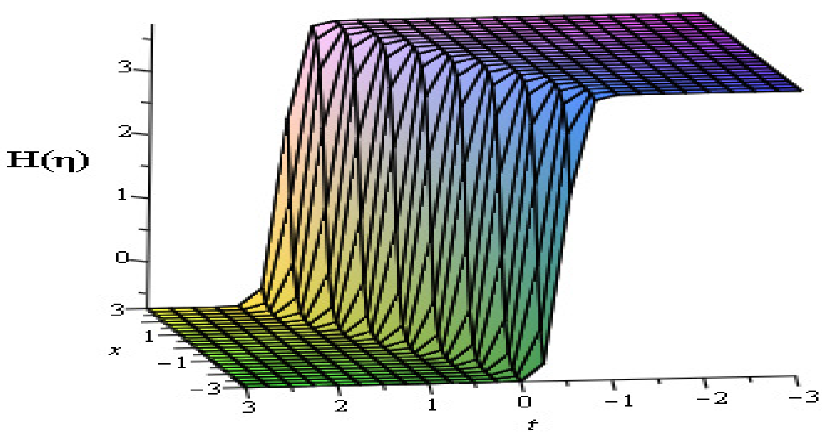

Fig.1(a): Plot of $H_1(\eta)$ for $A_0 = 4, E = 1, \lambda = 5, \mu = 1.25, y = 0$

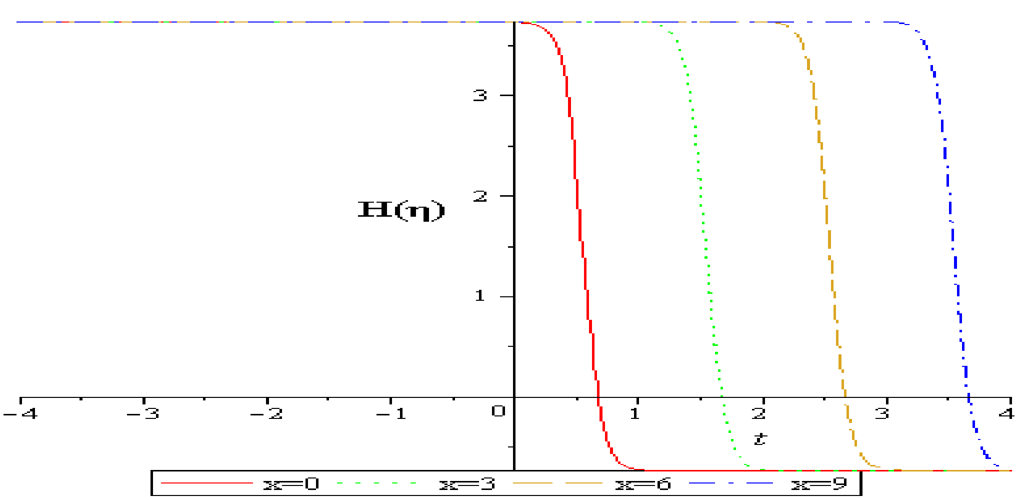

Fig.1(b): Plot of $H_1(\eta)$ wave against t for different values of x

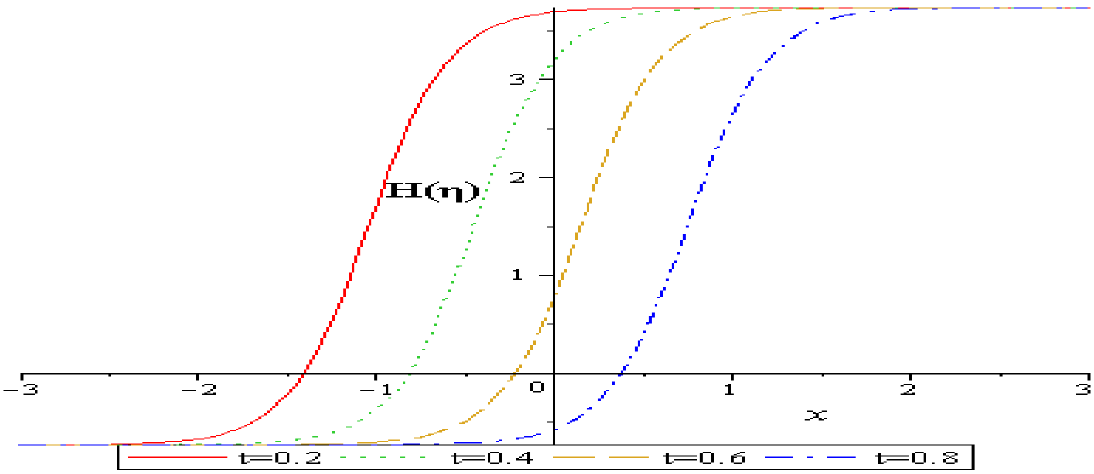

Fig.1(c): Plot of $H_1(\eta)$ wave against x for different values of t

The figure of the solutions $G_1(\eta)$ is similar to the figure of the solution $H_1(\eta)$.

Periodic solutions are traveling wave solutions that are periodic such as cos(x–t). Solutions $G_2(\eta)$ and $H_2(\eta)$ represent exact periodic traveling wave solutions of periodic wave and 3D profile of the solution $H_2(\eta)$ is given by the Fig. 2(a). Here the figure is obtained for cases $\lambda = 1, \mu = 1$ such that $\lambda^2 - 4\mu < 0$ and $A_0 = 1, E = 1$. Numerical representation produces that the same behavior as wave solutions. Time evolution of $H_2(\eta)$ wave for different values of displacement on the domain [-10, 10] are shown in 2D profile of Fig. 2(b) and we see here, $H_2(\eta)$ wave varies with displacement. It is found that the wave flow oscillates regularly that is periodic over the displacement region $-1 \le x \le 2$. In Fig. 2(c), we see $H_2(\eta)$ wave for different values of time t for the whole region of displacement $-3 \le x \le 3$ and time $0 \le t \le 3$. It is seen that the wave increases gradually as time increases.

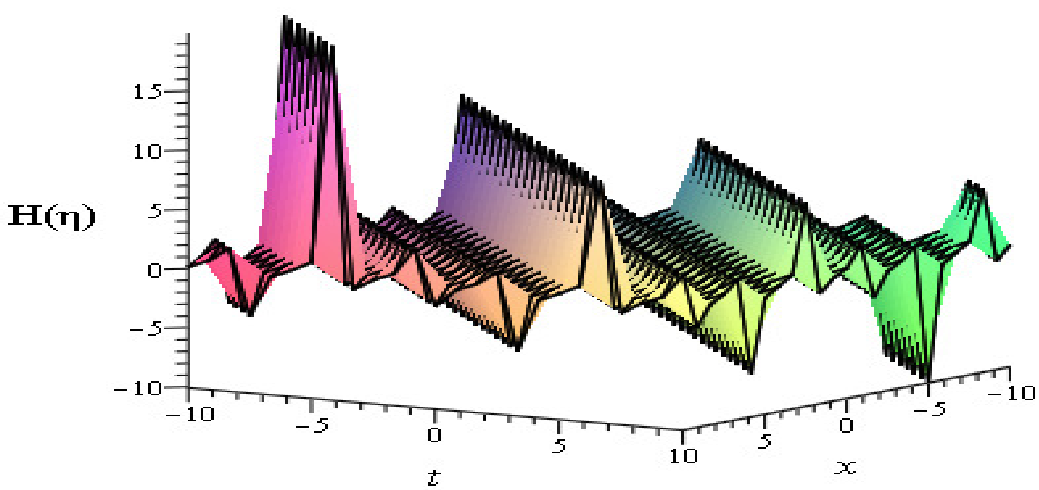

Fig.2(a): Plot of $H_2(\eta)$ for $A_0 = 1, E = 1, \lambda = 1, \mu = 1, y = 0$

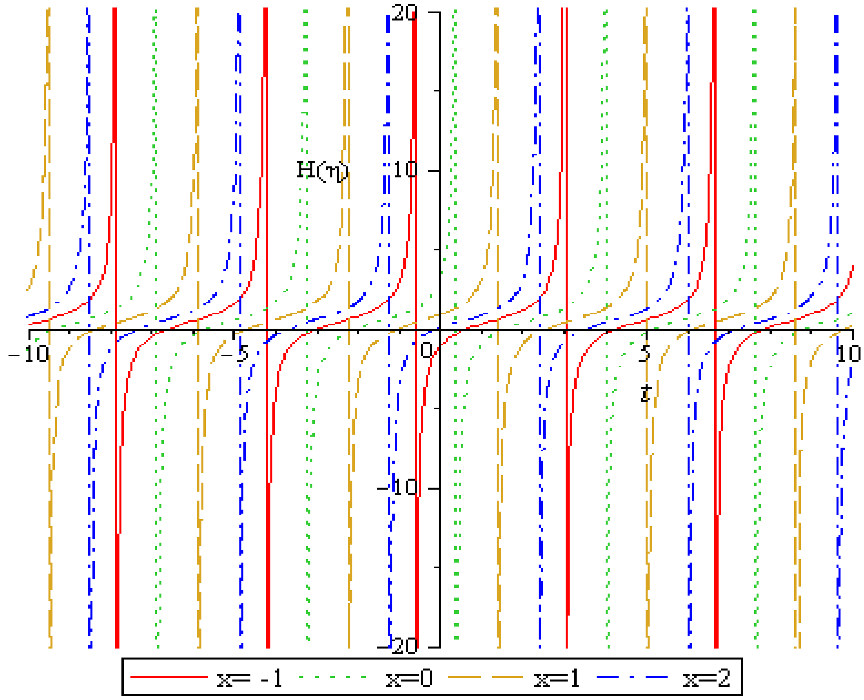

Fig.2-b: Plot of $H_2(\eta)$ wave against t for different values of x

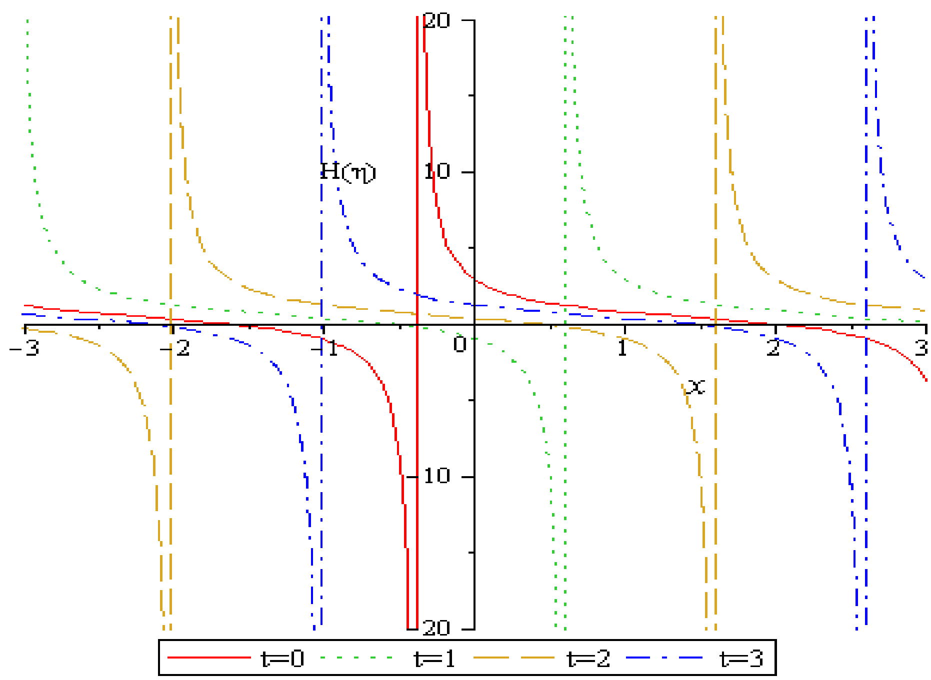

Fig.2(c): Plot of $H_2(\eta)$ wave against x for different values of t

The figure of the solutions $G_2(\eta)$ is similar to the figure of the solution $H_2(\eta)$.

Solitons are special kinds of solitary waves. The soliton solution is a specially localized solution. Hence $H'(\eta),\ G'(\eta),\ H''(\eta), G''(\eta) \to 0$ as $\eta \to \pm\infty,\ \eta = x + y - wt$. Solitons have a remarkable property that it keeps its identity upon interacting with other solitons. Solutions $H_5(\eta)$ represent singular Kink solution. 3D profile of the solution $H_5(\eta)$ is given in the Fig. 3(a). Here the figure is obtained for cases $\lambda = 0, \mu = 0$ such that $\lambda^2 - 4\mu = 0$ and $A_0 = 1, E = 1$. In 2D profile of Fig. 3(b) that gives the time evolution of $H_5(\eta)$ wave for different values of displacement on the domain [0, 3] (see figure for t from 0 to 3 only), we see that $H_5(\eta)$ wave varies with displacement. It is found that the wave flow oscillates regularly that is periodic over the displacement region $0 \le x \le 3$ and wave height also increases with time. Fig. 3(c) shows the $H_5(\eta)$ wave for different values of time t for the whole region of displacement $-10 \le x \le 10$ and time $0 \le t \le 3$. It is seen that the wave increases gradually as time increases and wave height also increases.

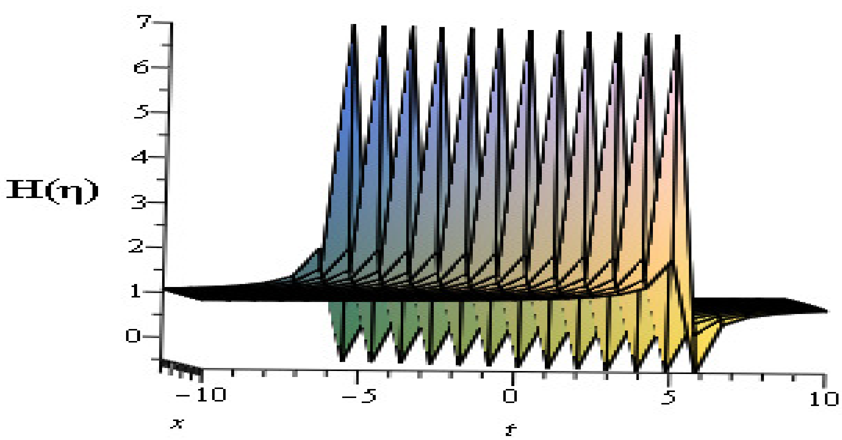

Fig. 3(a): Plot of $H_5(\eta)$ for $A_0 = 1, E = 1, \lambda = \mu = 0, y = 0$

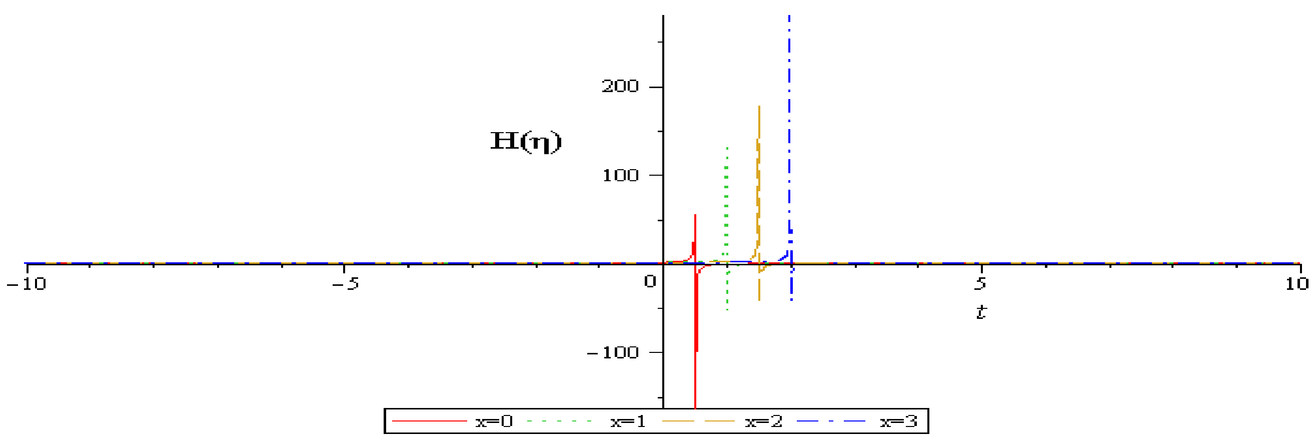

Fig. 3(b): Plot of $H_5(\eta)$ wave against t for different values of x

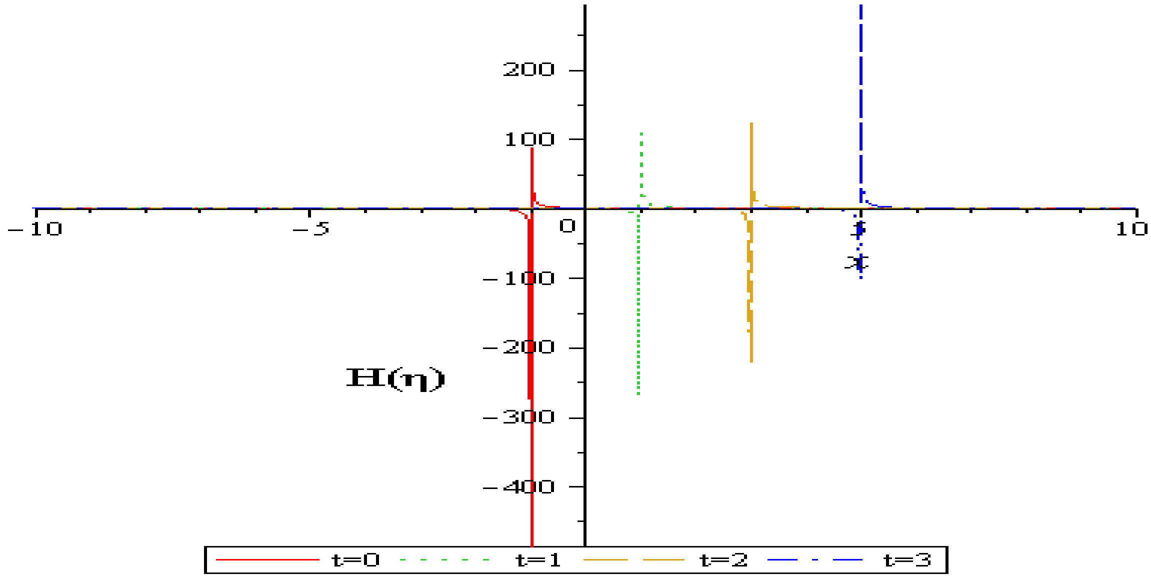

Fig. 3(c): Plot of $H_5(\eta)$ wave against x for different values of t

The figures of the solutions $H_3(\eta), G_3(\eta)$ and $G_5(\eta)$ are similar to the figure of the solution $H_5(\eta)$. Solutions $G_5(\eta)$ also dark soliton solution and 3D profile of the solution $G_5(\eta)$ are given in the Fig. 4(a). Here the figure is obtained cases for $\lambda = 0, \mu = 0$ such that $\lambda^2 - 4\mu = 0$ and $A_0 = 1, E = 1$. In 2D profile of Fig.4(b) that gives time evolution of $G_5(\eta)$ wave for different values of displacement on the domain [0, 10], we see $G_5(\eta)$ wave varies with displacement. Fig. 4(c) shows the $G_5(\eta)$ wave for different values of time t for the whole region of displacement $-8 \le x \le 8$ and time $0 \le t \le 3$. It is seen that the wave decreases gradually as time increases and wave height also increases.

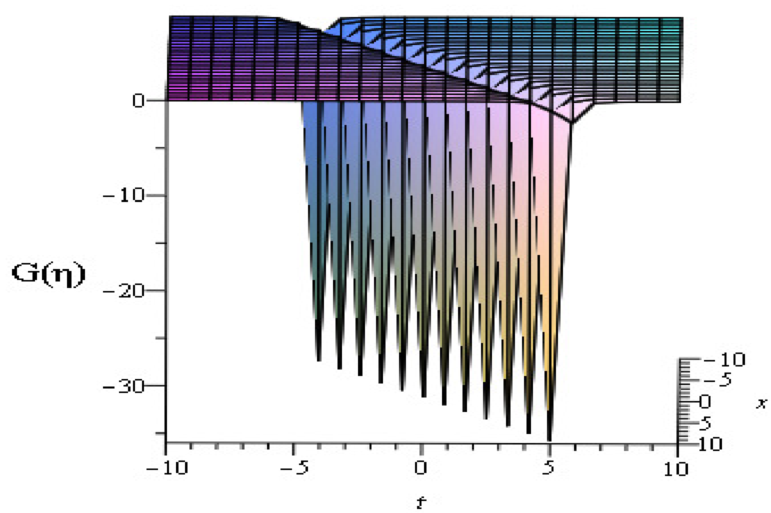

Fig 4(a): Plot of $G_5(\eta)$ for $A_0 = 1, E = 1, \lambda = \mu = 0, y = 0$

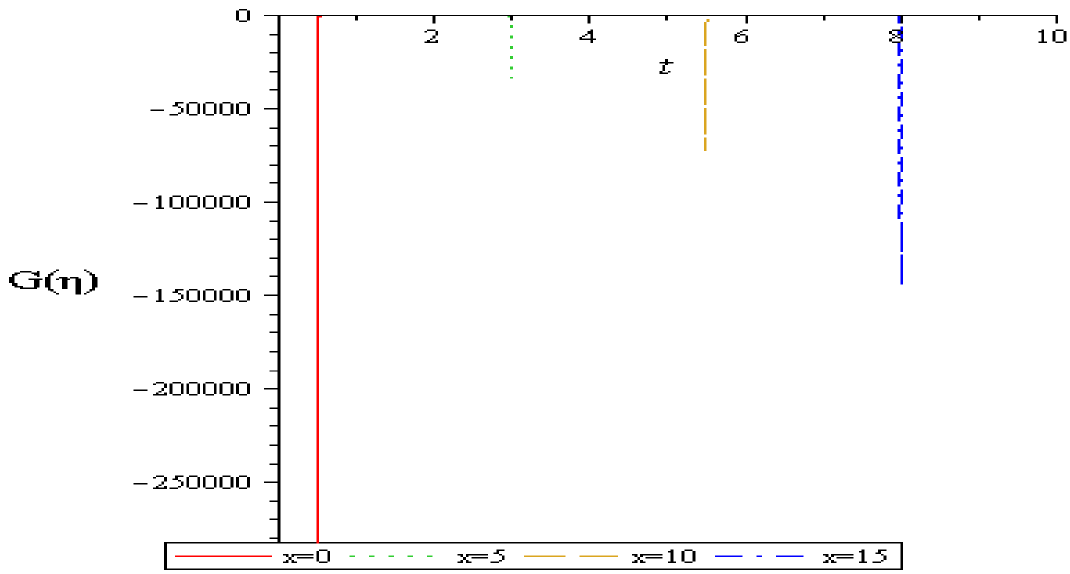

Fig.4(b): Plot of $G_5(\eta)$ wave against t for different values of x

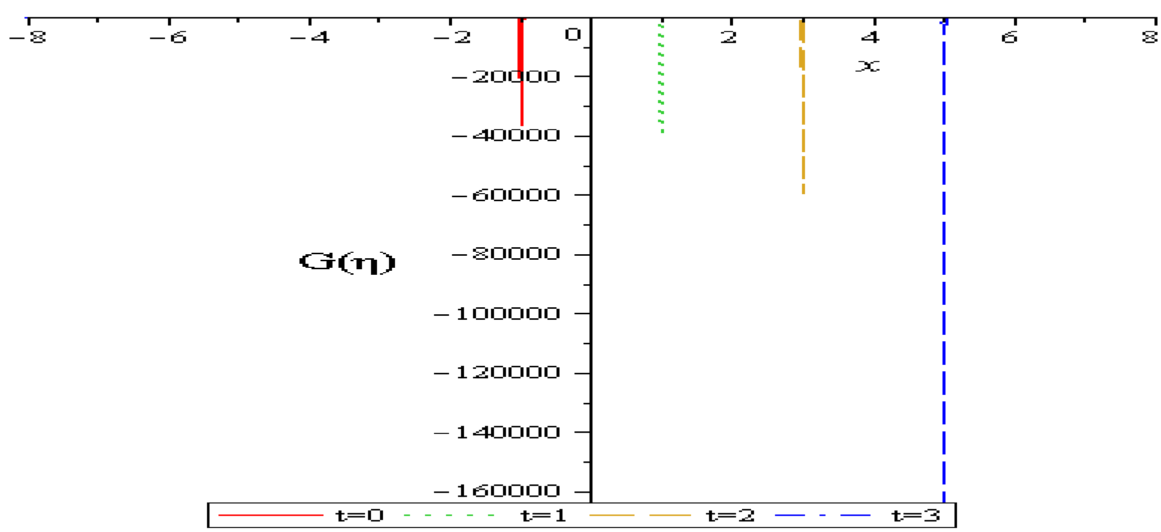

Fig.4(c): Plot of $G_5(\eta)$ wave against x for different values of t

5. Conclusion

In this work, we obtained new and exact solutions of the (2+1)-Dimensional Broer-Kaup equations by using the $\exp(-\phi(\eta))$ -expansion method. Results of the paper indicate that accurate soliton solutions obtained using $\exp(-\phi(\eta))$ -expansion method has large numbers of applications. Complicated physical phenomena in nonlinear model systems may be studied well in the future by applying the typical interaction solutions we have. The method may also be applied to other nonlinear partial differential equations (NPDE).

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