

A Non-causal Structure Mapping of Quantum State Space

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Abstract

The concept of infinity long ago led to an understanding that proper subsets of an infinite set can be put in a one-to-one correspondence with the original set. This mechanism of ‘parts’ of a set ‘duplicating’ the original set took on deeper significance when in 1914 Hausdorff showed that the three-dimensional sphere S^2 can be decomposed into a small finite number of pairwise disjoint subsets which may then be transformed by simple rotations to form two identical copies of S^2 . Further, the basic Hausdorff result may be extended to the case of the formation of any finite number of identical copies, and even to the case of countably infinite copies, if the restriction to finitely many initial pairwise disjoint pieces is relaxed to allow countably many such pieces.

We demonstrate the non-causal mechanics of state space multiplicity by the use of a free group of an independent pair of rotations acting on the state space of a qubit, S^2 , i.e. rotations from the group $SO(3)$. We go on to speculate on the ontological implications of these rotational actions, namely that quantum state space is the ontological basis of a quantum system, and admits non-causal non-classical structure mappings which may provide a framework for superposition basis state co-existence, and even for a frequentist probability theory.

1. Introduction

In 1914, Hausdorff published ‘Bemerkung u“ber den Inhalt von Punktmengen’ in *Mathematische Annalen* and included it in his book *Grundzu“ge der Mengenlehre* [1]. The paper contains a theorem which states that S^2 (the surface of a 3D ball) can be decomposed into finitely many pairwise disjoint subsets which can be rotated and formed into two identical instances of S^2 . This ‘Hausdorff Theorem’ was a striking result which challenged intuitive notions of measure and surface area, and preceded the more famous ‘Banach-Tarski Theorem’ (1924) involving solid 3D balls and volumes. Hausdorff’s insights are critical in understanding how Lebesgue non-measurable sets can behave with unexpected consequences under rotational group actions.

The partition and duplication of S^2 presents an intriguing framework for understanding the multiple instances of the state space S^2 , which do not coincide, and exist as distinct instances by necessity, and not by any natural cause. There may be many events which occur and suggest causation such as entanglement with another quantum system such as ‘measure’, but the multiplicity occurs by itself without cause. It is a non-causal, non-classical, non-intuitive map of state space to itself, with significant possible implications.

2. Main description

Let S^2 denote the unit sphere in R^3 , and let SO_3 denote the special orthogonal group in three dimensions, the group of all possible rotations about the origin in three-dimensional Euclidean space. SO_3 consists of all 3×3 matrices that preserve distances and orientations in 3D space under rotation. These matrices have two key properties: (a) Orthogonality: matrix M in SO_3 satisfies $M^{-1} = M^T$. (b) $\det(M)=1$: rotations preserve orientation whereas reflections reverse orientation and have determinant -1.

A free group of rank 2, denoted by F_2 , is the group generated by two symbols, say $\{x, y\}$, which consists of all possible finite sequences (‘words’) formed by concatenating the generators $\{x, y\}$ and their inverses $\{x^{-1}, y^{-1}\}$. It is assumed that all words in F_2 are ‘reduced’ meaning that they do not contain any concatenated pairs consisting of an element and its inverse. The group operation is left-concatenation followed by possible reduction. All free groups of the same rank are isomorphic so it is common to speak of ‘the’ free group of rank two F_2 .

The proof of the Hausdorff Theorem given below roughly follows Weston [3, sections 3-4] and Wagon [4, chapters 1-3]. First recall that a group G acts on a set X if to each $g \in G$ there corresponds a bijection $X \rightarrow X$, also denoted by g , such that for all $g, h \in G$ and $x \in X$, $g(h(x)) = (gh)(x)$, and $e(x) = x$ where e is the identity element of G . We begin by defining the key concept underlying all duplicative partitions of geometric spaces.

Definition 1: Let G be a group acting on a set X , and suppose E is contained in X . E is 'G-paradoxical' if for some natural numbers n and m there are pairwise-disjoint subsets of E $A_1, \dots, A_n$ and $B_1, \dots, B_m$, and elements of G $g_1, \dots, g_n$ and $h_1, \dots, h_m$ such that

$$E = \bigcup_{i \leq n} g_i A_i \text{ and } E = \bigcup_{j \leq m} h_j B_j$$

Then the Hausdorff Theorem may be stated as follows:

Theorem 1: There is a countable subset E of S^2 such that $S^2 - E$ is SO_3 -paradoxical. (Note that a more elaborate but inelegant version of the proof absorbs E into one of the other partitions yielding a simpler statement that S^2 is SO_3 -paradoxical. However in the present context of quantum mechanics, since the countable set E has measure zero, it can be disregarded from a probabilistic perspective.)

The Theorem defines a partition of S^2 into Lebesgue non-measurable sets via use of the Axiom of Choice (AC) (the role and significance of AC is discussed in section 4.) First it is shown that the free group F_2 is F_2 -paradoxical. Then a pair of independent rotations is defined in SO_3 which generate a free group of rank two in SO_3 . It is then shown that the free group in SO_3 can be used to partition S^2 into Lebesgue non-measurable sets A, B, C and D , and countable set E such that there exist rotations g and h in the free group contained in SO_3 with $S^2 - E = A \sqcup gB$, and $S^2 - E = C \sqcup hD$, and the unions are disjoint.

Note that $S^2 - E$ (or S^2) being SO_3 -paradoxical' is a denotative term and does not imply that the duplicative partition and re-assembly of subsets of S^2 is a mathematical paradox in which a statement contradicts itself or leads to an inconsistency. Furthermore, since the duplicative mechanics of the Hausdorff Theorem may be at the heart of the 'relative state' entanglement of quantum systems with superpositions, the methodology of the theorem may offer a new ontological intuition. (more discussion follows in section 4)

Let us begin the proof of Theorem 1 (Hausdorff Theorem). We begin by examining the free group F_2 of rank 2. This free group will play a key role in the methodology of the proof. Define $W(x)$ to be all reduced words in F_2 which begin with x . Define $W(y)$, $W(x^{-1})$ and $W(y^{-1})$ similarly.

(1) $F_2 = e \sqcup W(x) \sqcup W(y) \sqcup W(x^{-1}) \sqcup W(y^{-1})$ where e is the 'empty word' identity of F_2 . The unions are pairwise disjoint.

Now consider the set $x^{-1}W(x)$, all words of $W(x)$ left-concatenated by x^{-1} .

Lemma 1: $x^{-1}W(x) = e \sqcup W(x) \sqcup W(y) \sqcup W(y^{-1})$

Let $w \in W(x)$. Then w is a reduced word beginning with x so w cannot begin xx^{-1} , so $x^{-1}w$ cannot begin with x^{-1} . Therefore $x^{-1}W(x) \cap W(x^{-1}) = \emptyset$.

(2) Then by (1), $x^{-1}W(x) \subseteq e \sqcup W(x) \sqcup W(y) \sqcup W(y^{-1})$ Continuing, let $v = e \sqcup W(x) \sqcup W(y) \sqcup W(y^{-1})$

(a) If $v=e$, $v \in x^{-1}W(x)$ so $x^{-1}W(x)$ contains e .

(b) Suppose $v \in W(y)$. Then v begins with y , so $xv \in W(x)$. Then since $v = x^{-1}xv \in x^{-1}W(x)$, $x^{-1}W(x)$ contains $W(y)$.

(c) Next suppose $v \in W(x)$ then $xv \in W(x)$ also. Then since $v=x^{-1}xv \in x^{-1}W(x)$, $x^{-1}W(x)$ contains $W(x)$.

(d) finally suppose $v \in W(y^{-1})$. Then v begins with y^{-1} so $xv \in W(x)$. Then since $v=x^{-1}xv \in x^{-1}W(x)$, $x^{-1}W(x)$ contains $W(y^{-1})$.

(3) By (a)-(d) $x^{-1}W(x)$ contains $e \sqcup W(x) \sqcup W(y) \sqcup W(y^{-1})$.

Then by (2) and (3) the lemma is proved.

In a completely parallel demonstration lemma 1 can also be established for $W(y)$, $W(x^{-1})$ and $W(y^{-1})$:

$$y^{-1}W(y) = e \sqcup W(y) \sqcup W(x) \sqcup W(x^{-1})$$

$$xW(x^{-1}) = e \sqcup W(x^{-1}) \sqcup W(y) \sqcup W(y^{-1}) \quad yW(y^{-1}) = e \sqcup W(y^{-1}) \sqcup W(x) \sqcup W(x^{-1}) \text{ and}$$

Theorem 2: F2 is F2-paradoxical. Thus two proper subsets of F2 can be operated on and re-assembled to produce two copies of F2:

$$F_2 = x^{-1}W(x) \sqcup W(x^{-1}), \text{ and}$$

$$F_2 = y^{-1}W(y) \sqcup W(y^{-1}).$$

By (1) $F_2 = e \sqcup W(x) \sqcup W(x^{-1}) \sqcup W(y) \sqcup W(y^{-1})$ By Lemma 1 $x^{-1}W(x) = e \sqcup W(x) \sqcup W(y) \sqcup W(y^{-1})$ Then $F_2 = x^{-1}W(x) \sqcup W(x^{-1})$

Similarly, by a Lemma 1 variant, $y^{-1}W(y) = e \sqcup W(x) \sqcup W(x^{-1}) \sqcup W(y)$ Then again by (1) $F_2 = y^{-1}W(y) \sqcup W(y^{-1})$.

Note that AC is not used in the proof of Theorem 2.

Note also that F2 is countable since it is the union of a countable collection of finite sets of words (the words of length n for all n .)

Next we define two independent rotations which generate a free group of rank 2 in SO_3 .

Let ϕ be defined by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & -2\sqrt{2}/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{bmatrix}$$

Let ψ be defined by

$$\begin{bmatrix} 1/3 & -2\sqrt{2}/3 & 0 \\ 2\sqrt{2}/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Let } \phi^{-1} = \phi^T \text{ and } \psi^{-1} = \psi^T$$

Note that ϕ is a counter-clockwise rotation by $\arccos(1/3)$ about the x -axis (pitch), and ψ is a counter-clockwise rotation by $\arccos(1/3)$ about the z -axis (roll). We claim that $F_2(\phi, \psi)$, defined

analogously to $F_2 = F_2(x, y)$, is a free group of rank 2 in SO_3 . We omit the lengthy and algebraically tedious proof details, but they can be found in Wagon [5, Theorem 2.1]. We will use the F_2 -paradoxical nature of $F_2(\phi, \psi)$ in SO_3 acting on S^2 to define a paradoxical partition and re-assembly of subsets of S^2 .

Definition 2: For $p \in S^2$ the orbit of p with respect to a group G acting on

S^2 is $\text{orbit}(p) = \{g(p) \mid g \in G\}$.

The action of $F_2(\phi, \psi)$ on S^2 induces an equivalence relation on S^2 where p, q

$\in S^2$ are equivalent, i.e. $p \sim q$, if $\exists \alpha \in F_2(\phi, \psi)$ such that $\alpha(p) = q$, i.e. p and q have the same orbit.

Since $F_2(\phi, \psi)$ is countable, and since $\text{orbit}(p) = \{\alpha(p) \mid \alpha \in F_2(\phi, \psi)\}$, $\text{orbit}(p)$ is also countable. Then because S^2 contains a continuum of points there must be uncountably many orbits.

Proposition 1: Consider the $F_2(\phi, \psi)$ -orbits of S^2 . Let p and q be points of

S^2 . If $\text{orbit}(p) \neq \text{orbit}(q)$ then $\text{orbit}(p) \cap \text{orbit}(q) = \emptyset$.

Assume $\text{orbit}(p) \neq \text{orbit}(q)$ but $\exists s \in \text{orbit}(p) \cap \text{orbit}(q)$. Then $\exists \alpha$ and β in $F_2(\phi, \psi)$ with $\alpha(s) = p$ and $\beta(s) = q$. Then $\beta\alpha^{-1}p = q$, so $\text{orbit}(p) = \text{orbit}(q)$ a contradiction.

It is at this point we use the Axiom of Choice (AC) to select a single point from each of the uncountable $F_2(\phi, \psi)$ -orbits. Call this uncountable choice set M .

From (1) we have $F_2(\phi, \psi) = e \sqcup W(\phi) \sqcup W(\phi^{-1}) \sqcup W(\psi) \sqcup W(\psi^{-1})$.

For any $\alpha \in F_2(\phi, \psi)$ we define the set $\alpha M = \{\alpha(p) \mid p \in M\}$. Then $F_2(\phi, \psi)M$

$= \{\alpha(p) \mid \alpha \in F_2(\phi, \psi) \text{ and } p \in M\}$ is S^2 since M contains a point in every

$F_2(\phi, \psi)$ -orbit. Then we have the following decomposition of S^2 :

$$(5) \quad F_2(\phi, \psi)M = eM \sqcup W(\phi)M \sqcup W(\phi^{-1})M \sqcup W(\psi)M \sqcup W(\psi^{-1})M.$$

However the unions are no longer pairwise disjoint. For example, if p is one of the poles of $S^2 \cap X$ -axis then $p \in W(\phi)M$ and $p \in W(\phi^{-1})M$.

We will remove a countable set E from S^2 in order to define a pairwise-disjoint decomposition of $S^2 - E$. Let $E = \{p \in S^2 \mid \exists \alpha \in F_2(\phi, \psi) - \{e\} \text{ with } \alpha(p) = p\}$. E consists of all points of S^2 lying on axes of rotation of elements of $F_2(\phi, \psi)$. Since $F_2(\phi, \psi)$ is countable and there are exactly two poles on each axis of rotation, E is countable.

We now consider the action of $F_2(\phi, \psi)$ on $S^2 - E$. First we show that the action maps to $S^2 - E$ so the action is well-defined. Assume $\exists p \in S^2 - E$ and a rotation α in $F_2(\phi, \psi)$ with $\alpha(p) \in E$. By the definition of $E \exists \sigma$ in $F_2(\phi, \psi) - \{e\}$ with $\sigma\alpha(p) = \alpha(p)$. Then $\alpha^{-1}\sigma\alpha(p) = \alpha^{-1}\alpha(p) = p$, so $p \in E$, a contradiction, so the action of $F_2(\phi, \psi)$ on $S^2 - E$ produces points in $S^2 - E$, so is well-defined.

Continuing, we again use AC to choose a single representative point in each orbit of the action of $F_2(\phi, \psi)$ on $S^2 - E$. Let M' be the set of chosen representatives. To complete the proof of Theorem

1, i.e. the Hausdorff Theorem for S^2-E , the partition of $F_2(\phi, \psi)$ given in (1) is shown to induce a similar partition in S^2-E . For this we need the following:

Proposition 2: Let α_1 and α_2 be distinct elements in $F_2(\phi, \psi)$. Then $\alpha_1 M' \cap \alpha_2 M' = \emptyset$

Assume α_1 and $\alpha_2 \in F_2(\phi, \psi)$ and $\alpha_1 M' \cap \alpha_2 M' \neq \emptyset$. Then there is a point p in S^2-E with p in $\alpha_1 M'$ and p in $\alpha_2 M'$. Then there are points p_1 and p_2 in M' such that $p = \alpha_1(p_1)$ and $p = \alpha_2(p_2)$. From $p = \alpha_2(p_2)$ we obtain $\alpha_1^{-1}p = \alpha_1^{-1}\alpha_2(p_2)$. Then $\alpha_1^{-1}(\alpha_1(p_1)) = \alpha_1^{-1}\alpha_2(p_2)$, thus $p_1 = \alpha^{-1}\alpha_2(p_2)$.

Therefore p_1 and p_2 belong to the same orbit.

Since M' contains a unique representative from each orbit, $p_1 = p_2$. But then p_1 is a fixed point of $\alpha^{-1}\alpha^2$.

However, since $p_1 \in M'$, $p_1 \in S^2-E$, but then to be a fixed point, $\alpha_1^{-1}\alpha_2 = e$,

so $\alpha_1 = \alpha_2$.

Thus if $\alpha_1 M' \cap \alpha_2 M' \neq \emptyset$, $\alpha_1 = \alpha_2$, so $(\alpha_1 \neq \alpha_2) \rightarrow (\alpha_1 M' \cap \alpha_2 M') = \emptyset$.

By Proposition 1, the partition given in (1) denoted in terms of $F_2(\phi, \psi)$ is:

$$F_2(\phi, \psi) = e \sqcup W(\phi) \sqcup W(\phi^{-1}) \sqcup W(\psi) \sqcup W(\psi^{-1}).$$

$$\text{Therefore } S^2-E = F_2(\phi, \psi) M' = eM' \sqcup W(\phi)M' \sqcup W(\phi^{-1})M' \sqcup W(\psi)M' \sqcup W(\psi^{-1})M'.$$

Now we can apply Theorem 2 to induce a $F(\phi, \psi)$ -paradoxical partition of S^2-E :

$$S^2 - E = \phi^{-1}(W(\phi)M') \sqcup W(\phi^{-1})M'$$

$$S^2 - E = \psi^{-1}(W(\psi)M') \sqcup W(\psi^{-1})M'$$

which is the Hausdorff Theorem for S^2-E

3. Discussion

The demonstration given in section 3 is the simplest multiplying state space scenario in the sense that it requires only four pairwise disjoint pieces, two per instance, which when rotated form two identical instances of S^2-E where E is a countable set (and hence measure zero, and probability zero), and by minor adjustment to the proof, two identical instances of S^2 , the state space itself. There are three key points associated with the demonstration which prompt further discussion.

The first is that the demonstration strongly suggests that the 'ontic identity' of a quantum system is its state space, since the space has a more fundamental role in structural mappings than the state vectors traversing or existing within the space. Each point of the state space corresponds to a possible pure state of the system, and unitary actions via rotation are sufficient to map the spatial structure to multiple instances, independent of the assumed position of the state vector within

each instance. The formation of multiple state space instances suggests that the underlying reality admits non-causal, non-classical and non-intuitive structure mappings at a level of hierarchy above that of the state vector. These instances are real aspects of a larger ontological domain than previously considered by quantum physics, and may provide a ground for multiple co-existing superposition basis states which does not 'occur' by any causal means, but exists by mathematical necessity alone.

Mathematical necessity brings up a second point of discussion - the role of the Axiom of Choice (AC) in the demonstration of section 3. AC states that given any collection of non-empty sets, one can choose a single member from each set, or in other words, there is a 'choice function' which selects a single element from each set in the collection. I think it is fair to say that the axiom itself seems completely plausible, and that the vast majority of mathematicians accept it, especially since it is required in the proof of countless fundamental mathematical results such as the following - that infinite cardinals are comparable; that every infinite set has a countable subset; Zorn's Lemma and the Well Ordering Theorem (equivalent to AC), and many more.

If it is controversial at all, it is because it is non-constructive - it asserts the existence of the choice function without requiring an explicit definition. Its place in the axiomatic foundation of mathematics is quite solid. The axioms of what is termed 'Zermelo-Fraenkel' (ZF) set theory are considered the bedrock of mathematical thought. AC is independent of those axioms, so it is certainly not disproved by them. However, the vast majority of mathematicians accept Axiom of Choice as an additional bedrock axiom, and hence the foundation of current mathematics (and hence physics) is ZF+AC, referred to as ZFC. All of this discussion is to reinforce acceptance of the validity of the section 3 demonstration. It can be added that much of the Functional Analysis Theory underlying Quantum Mechanics and Quantum Field Theory depends on AC, mainly in the form of Zorn's Lemma, invoked when a maximal or exhaustive choice is needed from an infinite set. For a discussion of many perplexing difficulties for mathematics, and hence physics, after rejection of AC see Jech [7, chapter 10].

A third point is the generalization of the demonstration of section 3, or some variation or new demonstration, for arbitrary quantum systems and corresponding state spaces. It is trivial to extend the demonstration of section 3 to the formation of k instances of the state space for all natural numbers k with $k > 2$, and for $k = \omega$. Simply iterate the demonstration for one instance of S^2 to as many as needed by induction. A more elegant argument exists in which there are specific free group collections and rotations which produce the formation of k instances from finitely many pieces for k any natural number greater than 2, and for the case of $k = \omega$ if the finitely many pieces restriction in the demonstration is relaxed to allow a countable set of pairwise disjoint pieces. However, the same argument cannot be generalized to an uncountable set of instances of S^2 . However, a somewhat different argument is given in Wagon [4, chapter 6] which demonstrates the case for multiplicity of S^2 for the case of a continuum of instances.

4. Conclusion

We have demonstrated the non-causal mechanics of state space multiplicity by the use of a free group of an independent pair of rotations acting on the state space of a qubit, S^2 , i.e. rotations from the group $SO(3)$. We have speculated on the ontological implications of these rotational actions, namely that

quantum state space is the ontological basis of a quantum system, and its state space admits non-causal non-classical structure mappings which may provide a framework for superposition basis state co-existence, and even for a frequentist probability theory.

It is beyond the scope of this paper to develop non-causal state space structure mappings for all quantum systems. However, since the pure state spaces of finite and separable infinite dimensional Hilbert spaces admit unitary actions of the free group of rank 2, it seems likely that some generalized argument to that given in section 3 would yield similar results.

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