Dirac Like Equation for Free Particles of Any Spin

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Abstract

Adopting different matrix representations of the Clifford Cl3 algebra, a Dirac like equation provides a general relativistic wave equation that can describe particles of any spin. In analogy with the Dirac spinors, the wave functions split into two spinors, one accounting for a particle with a definite four momentum and one designating a helicity Eigen-state. Rewritten in spherical coordinates, generic solutions for any spin are presented.

Keywords: Clifford Cl3 algebra, Dirac spinors, Relativistic wave equation

The problem of generalizing the Dirac equation to account for free massive particle of any spin has a long history [1-4] (For a historic review of such attempts we address the reader to ref. [3-4] and references mentioned therein). The Dirac equation [1] in its standard form is,

$$\left(i\gamma^{\mu}\partial_{\mu} - M\right)\Phi = 0, \tag{1}$$

Where γ^{μ} are the Dirac 4×4 gamma matrices, satisfying,

$$(\gamma^0)^2 = I^{(4)}\ ;\ (\gamma^{\mu})^2 = -1; \gamma^{\mu}\gamma^{\nu} = -\gamma^{\nu}\gamma^{\mu}\ \ ;\ \mu,\nu = 1,2,3. \tag{2}$$

It is typical to the Clifford algebra *Cl3* of Euclidean space that it can be generated by eight elements [5],

$$\text{A scalar : } 1, \tag{3a}$$

$$\text{Three vectors : } e_i; i = 1,2,3, \tag{3b}$$

$$\text{Three bivectors : } e_{ij} = e_i e_j = e_i \cdot e_j + e_i \wedge e_j, \tag{3c}$$

$$\text{One Trivector : } e_{123} = (e_1 e_2 e_3)(e_1 e_2 e_3) = -1. \tag{3d}$$

The Pauli matrices are isomorphic to *Cl3* providing its matrix representation. Namely $1 \leftrightarrow I^{(2)}$, $e_i \leftrightarrow \sigma_i$. Likewise, the Dirac matrices, spanned by $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ as orthonormal unit vectors parallel to the coordinate axes of Minkowski space time, with $(\gamma_0)^2 = 1, (\gamma_i)^2 = -1\ ; i = 1,2,3$ form also generators of the Clifford algebra. Since the 2×2 Pauli spin matrices and the 4×4 Dirac matrices form representations of *Cl3* all algebraic properties of Pauli spin matrices and the Dirac matrices follow from the underlying algebra. This observation must hold true for other representations and can be applied, as demonstrated below, to derive a Dirac like equation for particles of any spin. As a representation of *Cl3* algebra consider the following gamma matrices,

$$\gamma_i^{(2n)} \equiv diag(\sigma_i, \cdots, \sigma_i)\ ; i = x, y, z, \tag{4}$$

where $\sigma_i; i = x, y, z$ are the Pauli matrices. Clearly, the above satisfy,

$$(\gamma^\mu)^\dagger = \gamma^\mu;\ (\gamma^\mu)^2 = \gamma^0\ \ ; \gamma^0 = I^{(2n)}\ ; \mu = 0,1,2,3, \tag{5a}$$

$$[\gamma^a, \gamma^b] = i\epsilon^{abc}\gamma_c\ \ ; a, b, c = x, y, z, \tag{5b}$$

$$\{\gamma^a, \gamma^b\} = 2\gamma^0\delta^{ab}; a, b = x, y, z, \tag{5c}$$

$$trace\, \gamma^a = 0\ \ ; a, b, c = x, y, z, \tag{5d}$$

$$\gamma^a \beta - \beta \gamma^a = 0, \beta^2 = I^{(2n)}, \tag{5e}$$

where $I^{(n)}$ is $n \times n$ unit matrix and $\beta = diag(I^{(n)}, -I^{(n)})$. The gamma matrices defined above, like the Pauli spin matrices and the Dirac matrices, are isomorphic to Cl_3 and allow factorizing the momentum-energy relation as follows,

$$(E^2 - p^2 - M^2)I^{(2n)} = \left[E\, I^{(2n)} + \begin{pmatrix} MI^{(n)} & \boldsymbol{p}\cdot\boldsymbol{\gamma}^{(n)} \\ \boldsymbol{p}\cdot\boldsymbol{\gamma}^{(n)} & -MI^{(n)} \end{pmatrix}\right]\left[E\, I^{(2n)} - \begin{pmatrix} MI^{(n)} & \boldsymbol{p}\cdot\boldsymbol{\gamma}^{(n)} \\ \boldsymbol{p}\cdot\boldsymbol{\gamma}^{(n)} & -MI^{(n)} \end{pmatrix}\right] = 0, \tag{6}$$

and we may adopt the following as a Dirac like equation for any spin s,

$$\left[E\, I^{(2n)} - \begin{pmatrix} MI^{(n)} & \boldsymbol{p}\cdot\boldsymbol{\gamma}^{(n)} \\ \boldsymbol{p}\cdot\boldsymbol{\gamma}^{(n)} & -MI^{(n)} \end{pmatrix}\right]\Phi^{(2n)} = 0. \tag{7}$$

Above, the wave function $\Phi^{(2n)}(x,t)$ is $2n$ components spinor, $E, \boldsymbol{p}, M$ are respectively, the particle energy, linear momentum and rest mass. For reasoning to be given below, we identify the vector $\boldsymbol{\gamma} = (\gamma_x, \gamma_y, \gamma_z)$ with the particle spin $\boldsymbol{S}$. Taking, $n = 2s + 1$, $\beta \equiv diag(I^{(n)}, -I^{(n)})$, the Minkowski metric $\eta_{\mu\nu} = diag(1, -1, -1, -1)$, $E \leftrightarrow i\partial_t$ and $\boldsymbol{p} \leftrightarrow (-i\partial_x, -i\partial_y, -i\partial_z)$, Eq. (7) can be rewritten compactly as,

$$(i\gamma^\mu\eta_{\mu\nu}\partial_\nu - \beta M)\Phi^{(2n)} = 0. \tag{8}$$

Formally, apart from the gamma matrices being different the above has the form of Eq. (1) and is subjected to the *Cl3* algebra. Here as well the wave function can be considered as $2n$ column vector or as two n–components spinor wave functions, $\Phi^{(2n)} = \begin{pmatrix} \phi_1^{(n)} \\ \phi_2^{(n)} \end{pmatrix}$, where one of these accounts for a particle with four momentum $(E, \boldsymbol{p})$ while the other designates a helicity eigen-state. Taking $n = 2s +$

1 is just right to account for the allowed spin components $m_s = s, s-1, s-2, \cdots, -s+2, -s+1, -s$. For $s = 1/2$ fermion Eq. (8) with the gamma matrices of Eq. (4) yields exactly the same four Dirac Spinors.

As mentioned above we identify the gamma matrices with the spin components. Indeed, the commutative relations Eq. (5b) of the gamma matrices are reminiscent of the spin operator commutation relations,

$$[S_x\,, S_y] = iS_z, [S_z\,, S_x] = iS_y\,, [S_y\,, S_z] = iS_x\,. \tag{9}$$

Furthermore, neither the orbital angular momenta, $L_i \equiv -i\epsilon^{ijk}x_j\partial_k$, nor the spin components $s_i \equiv i\epsilon^{ijk}\gamma_j\partial_k$ commute with the free particle Hamiltonian of Eq. (8). It is straightforward to prove that,

$$[L_i, H_0] = +\epsilon^{ijk}\gamma_j\partial_k, \tag{10}$$

and,

$$[s_i, H_0] = -\epsilon^{ijk}\gamma_j\partial_k\,. \tag{11}$$

Then, $j_i = L_i + \gamma_i$ commutes with H_0 and the total angular momentum taken to be $\mathbf{J} = \boldsymbol{L} + \boldsymbol{S} = \boldsymbol{L} + \boldsymbol{\gamma}$ is conserved as expected for the total angular momentum. Naturally, since the free particle Hamiltonian Eq. (8) is $2n \times 2n$ the wave function $\Phi(\mathrm{x}, \mathrm{t})$ is a $2(2s+1)$ components spinor. This is just right to allow for $2(2s+1)$ orthonormal solutions, $2s+1$ positive energy and $2s+1$ negative energy solutions with helicity Eigen-values, $h = s, s-1, \cdots, -s+1, -s$. Particularly, for spin $s = 1/2$ one obtains four components spinors corresponding to 2 positive energy solutions with spin up and spin down and 2 negative energy solutions with spin up and spin down.

For practical purposes it is worth rewriting Eq. (8) in spherical coordinates. To this aim the Cartesian Minkowski metric is replaced with the Minkowski metric written in Spherical Coordinates,

$$i\gamma^0\partial_t\Phi^{(2n)} = (-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + \beta M)\Phi^{(2n)}, \tag{12}$$

where $\boldsymbol{\gamma} \equiv (\gamma^t, \gamma^r, \gamma^\vartheta, \gamma^\varphi)$ with $\gamma^t, \gamma^r, \gamma^\vartheta, \gamma^\varphi$ being related to the γ^μ; $\mu = 0,1,2,3$ via a similarity transformation,

$$\begin{pmatrix}\gamma^t\\ \gamma^r\\ \gamma^\vartheta\\ \gamma^\varphi\end{pmatrix} = \begin{pmatrix}E_0^0 & 0 & 0 & 0\\ 0 & E_1^1 & 0 & 0\\ 0 & 0 & E_2^2 & 0\\ 0 & 0 & 0 & E_3^3\end{pmatrix}\begin{pmatrix}\gamma^0\\ \gamma^1\\ \gamma^2\\ \gamma^3\end{pmatrix}. \tag{13}$$

Above, $E_0^0 = 1, E_1^1 = 1, E_2^2 = 1/r, E_3^3 = 1/r\sin\vartheta$ stand for inverse vierbein fields [see for example 6], and $\boldsymbol{\nabla} = \hat{r}\partial_r + \hat{\vartheta}\partial_\vartheta + \hat{\varphi}\partial_\varphi$. By definition the angular momentum operator is, $\hat{L} = \hat{r} \times \boldsymbol{\nabla}$. Then, $\hat{r} \times \hat{L} = i\,[\boldsymbol{\nabla} - \hat{r}\partial_r]$, and

$$i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} = i\boldsymbol{\gamma}\cdot\hat{r}\partial_r + \boldsymbol{\gamma}\cdot\hat{r}\times\hat{L}\,. \tag{14}$$

Using the identity: $(\boldsymbol{\gamma}\cdot \mathrm{A})(\boldsymbol{\gamma}\cdot \mathrm{B}) = (\mathrm{A}\cdot \mathrm{B}) + \frac{i}{2}\,\boldsymbol{\gamma}\cdot A\times B$ where A and B are any three vectors (Note that since the gamma matrices are representation of *Cl3* this identity follows from the underlying algebra) we may rewrite the free particle Hamiltonian as,

$$H_0 = i\gamma^0\partial_t = -i\boldsymbol{\gamma}\cdot\hat{r}\left[\partial_r + \frac{s}{r} - \frac{1}{r}(2\boldsymbol{\gamma}\cdot L + s)\right] + \beta M\ . \tag{15}$$

As mentioned above, we identify $\boldsymbol{\gamma}$ with the spin $\boldsymbol{S}$. This allows defining an angular operator $K \equiv (2\boldsymbol{\gamma}\cdot L + s) = (2\boldsymbol{S}\cdot L + s)$,

$$K = 2\boldsymbol{S}\cdot L + s = J^2 - L^2 - S^2 + s. \tag{16}$$

Substituting this in Eq. (15) gives,

$$H_0 = i\gamma^r\left[\partial_r + \frac{s}{r} - \frac{1}{r}K\right] + \beta M. \tag{17}$$

It is rather straightforward to show that the total angular momentum J^2, the third component of the total angular momentum J_3, the angular operator K and parity P commute with the Hamiltonian H_0, Eq. (17). Then, we may separate the wave function $\Phi^{(2n)}$ into angular and radial wave functions. The angular part of the wave function is a spinor of spherical harmonics. These are Eigen-states of J^2 and J_3 and parity. The Eigen functions of L^2 and L_3 are the usual spherical harmonics $Y_l^{m_l}(\Omega)$. The Eigen functions of S^2 and m_3 are $2s+1$ components spinors. For a spin s particle $m_s = -s, -s+1, \cdots, s-1, s$. The angular part of the wave functions for free particles are readily written as spherical harmonic spinors,

$$y^j_{lm_j}(\Omega) = \sum_{m_s=-s}^{m_s=+s} C(l,s,j;m_j-m_s,m_s,m_j)\,Y_l^{m_l}(\vartheta,\varphi)\chi_s^{m_s}. \tag{18}$$

Above, $C(l,s,j;m_j-m_s,m_s,m_j)$ is a Clebsch Gordon coefficient combining orbital angular momentum l with spin s to a total angular momentum j with magnetic quantum numbers $m_j - m_s, m_s, m_j$, respectively. The spherical harmonic spinors are orthonormal,

$$\int d\Omega\left(y^j_{lm_j}(\Omega)\right)^H y^{j'}_{l'm_j'}(\Omega) = \delta_{jj'}\delta_{ll'}\delta_{mm_j'}. \tag{19}$$

Furthermore, the wave function $\Phi^j_{m_j\kappa_j}$ is an Eigen-function of J^2, J_3, K and parity, thus

$$J^2\Phi^j_{m_j\kappa_j} = j(j+1)\Phi^j_{m_j\kappa_j}, \tag{20}$$

$$J_3\Phi^j_{m_j\kappa_j} = m_j\Phi^j_{m_j\kappa_j}, \tag{21}$$

$$K\Phi^j_{m_j\kappa_j} = \kappa_j\Phi^j_{m_j\kappa_j}. \tag{22}$$

The eigenvalues κ_j of the angular operator K for $j = l \pm s$ are determined by evaluating the eigenvalues of $K^2 = (J^2 - L^2 - S^2 + s)^2$, one finds,

$$(\kappa_j)^2 = s^2(2l+1)^2. \tag{23}$$

$$\kappa_j = \pm s(2l+1). \tag{24}$$

Thus, assuming $\Phi^{j}_{m_j \mathrm{k}_j} \sim \exp(-i\omega t)$, for both $j = l + s$ and $j = l - s$ there exist two solutions $R_1(r)\phi^{+}(\Omega)\exp(-i\omega t)$ and $R_2(r)\phi^{-}(\Omega)\exp(-i\omega t)$ corresponding to positive and negative κ_j respectively.

Consider now the radial wave functions $R_1(r)$ and $R_2(r)$. With the free particle Hamiltonian Eq. (17) these are solutions of the equations,

$$H_0 \Phi^{j}_{m_j \mathrm{k}_j} = i\gamma^0 \partial_t \begin{pmatrix} R_1(r)\phi^{+}(\Omega) \\ R_2(r)\phi^{-}(\Omega) \end{pmatrix} = \left(-i\gamma^r \left[\partial_r + \frac{s}{r} - \frac{1}{r}K\right] + \beta M\right) \begin{pmatrix} R_1(r)\phi^{+}(\Omega) \\ R_2(r)\phi^{-}(\Omega) \end{pmatrix}. \tag{25}$$

Substituting $\gamma^r = diag(\sigma^1, \cdots, \sigma^1)$, Eq. (25) becomes equivalent to $n/2$ identical non-autonomous pairs of equations,

$$\begin{pmatrix} (\omega - M)R_1(r)\phi^{+}(\Omega) \\ (\omega + M)R_2(r)\phi^{-}(\Omega) \end{pmatrix} = -i\sigma^1 \begin{pmatrix} \left[\frac{d}{dr} + \frac{s-\kappa_j}{r} R_1(r)\phi^{+}(\Omega)\right] \\ \left[\frac{d}{dr} + \frac{s+\kappa_j}{r} R_2(r)\phi^{-}(\Omega)\right] \end{pmatrix}. \tag{26}$$

These we may rewrite as pairs of autonomous radial equations:

$$\left[\frac{d^2}{dr^2} + \frac{2s}{r}\frac{d}{dr} + \frac{(\kappa_j - 1)\kappa_j + (s-1)s}{r^2} + (\omega^2 - M^2)\right] R_1(r) = 0, \tag{27}$$

$$\left[\frac{d^2}{dr^2} + \frac{2s}{r}\frac{d}{dr} - \frac{\kappa_j(\kappa_j + 1) - (s-1)s}{r^2} + (\omega^2 - M^2)\right] R_2(r) = 0. \tag{28}$$

Furthermore, the equations above can be rearranged as a transformed Bowman version of the Bessel differential equation [7], namely,

$$\left[r^2 \frac{d^2}{dr^2} + (2p+1) r \frac{d}{dr} + \left(\alpha^2 r^2 + {\beta_1}^2\right)\right] R_1 = 0, \tag{32}$$

$$\left[r^2 \frac{d^2}{dr^2} + (2p+1) r \frac{d}{dr} + \left(\alpha^2 r^2 + {\beta_2}^2\right)\right] R_2 = 0, \tag{33}$$

where $2p + 1 = 2s$, ${\beta_1}^2 = (\mathrm{k}_j - 1)\mathrm{k}_j + (s-1)s$, ${\beta_2}^2 = \mathrm{k}_j(\mathrm{k}_j + 1) - (s-1)s$ and $\alpha^2 = (\omega^2 - M^2)$. In terms of the first and second Bessel functions, the radial wave functions for free particle of any spin are given by,

$$R_i(r) = r^{-p}\left[C_1 J_{q_i}(\alpha r) + C_2 Y_{q_i}(\alpha r)\right] ; i = 1,2, \tag{34}$$

where $q_i = \sqrt{p^2 - {\beta_i}^2}$, and C_i are constants. The quantities p and β_i both depend on the spin, so that the above represent implicit generic solutions for any spin. We may conclude that, in a stationary flat Minkowski space-time, the wave function of a free particle of any spin has the form,

$$\Phi^{j}_{m_j \mathrm{k}_j}(t, r, \Omega) = e^{(-i\omega t)} \left[A_{j,m_j,\kappa_j}(p) R_{1,\kappa_j}(r)\phi^{+}(\Omega) + B_{j,m_j,\kappa_j}(p) R_{2,\kappa_j}(r)\phi^{-}(\Omega)\right] \times y^{j}_{l m_j}(\Omega), \tag{35}$$

where, $A_{j,m_j,\kappa}$ and $B_{j,m_j,\kappa}$ are complex coefficients depending on the particle momentum.

To conclude, we quote without proves the following: (1) Equation (8) is fully consistent with Quantum Mechanics, and can be derived from a Lagrangian density and by using the Noether's Theorem the conserved current satisfies the continuity equation. (2) Second quantization can be accomplished using n particle and n anti-particle positive energy states. All be it Eq. (8) furnishes free particle relativistic dynamics. (3) In the limit of $M \to 0$ the equation reduces to that corresponding to massless particles which was studied thoroughly in a global space-time [8]. (4) As a private case, it is to be noted, that for massive $s = 1$ bosons Eq. (8) is equivalent to the Proca equations [9] and in the limit of vanishing mass to the Maxwell's equations. Proves of these will be reported in future report.

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