Analytical solutions of the null-geodesics in Ellis-Bronnikov wormhole spacetime via $\exp(-\Phi(\xi))$-expansion method

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Abstract

In this article, we use the so-called $\exp(-\Phi(\xi))$ -expansion method to obtain some specific classes of one-parameter exact solutions of null-geodesics in the Ellis- Bronnikov wormhole metric. In the first stage of this method, the nonlinear PDE is converted into a nonlinear ordinary derivative equation (ODE) of polynomial form. Therefore, if we initially have a nonlinear ODE of polynomial form, sometimes its solutions can be obtained using the procedure of the $\exp(-\Phi(\xi))$-expansion method. In our paper, this method allows us to obtain some exact analytical solutions to null-geodesic equations in the Ellis-Bronnikov wormhole metric, expressed in the elementary functions.

Keywords: Ellis-Bronnikov wormhole, Null-Geodesic Equation, Exact Solution, $\exp(-\Phi(\xi))$-Expansion Method.

1. Introduction

The theoretical studies of space-time structures with non-trivial topology have led to the discovery of some hypothetical objects, the so-called wormholes [1, 2] . The earliest traversable wormhole was proposed independently by Ellis and Bronnikov [3, 4] and is now known as the Ellis-Bronnikov wormhole, which is the simplest Morris-Thorne wormhole [5, 6]. Later, for the Ellis-Bronnikov wormhole metric, as the simplest traversable wormhole metric, different generalizations have been obtained generated by several kinds of exotic matter (see, e.g., [7]-[9]). Moreover, many papers have been devoted to many studies of null and time-like geodesics in the gravitational fields of various wormholes.

Since the equations of motion in different wormhole geometries are mostly of elliptical types, their solutions can be expressed through the Weierstraß functions [10]-[14] . Note that quite often null and timelike geodesic equations are preferred to be solved numerically, representing the corresponding trajectories and geodesic potentials graphically (see, for example, [15] and references there).

To find the solitary wave and soliton solutions of non-linear evolution equations or non-linear partial differential equations, many expansion methods have been proposed, which have received considerable attention in recent decades. Such methods include, for example, the homogeneous balance method [16, 17] , the tanh-function method [18] , the sine-cosine method [19] , the $\exp(-\Phi(\xi))$ -expansion method [20] , the F-expansion method [21] , the Jacobi elliptic function method [22]-[24] , the (G'/G) -expansion method [25]-[27], and the wide variety of their modifications.

These methods have been developed in order to construct exact analytical solutions of the different nonlinear evolution equations and nonlinear partial derivative equations (PDE). The interesting feature of those methods is that at the first stage of solving PDE, the independent space and time variables are combined into one traveling wave variable. As a result, the initial PDE is transformed into corresponding nonlinear ordinary derivative equation (ODE). If this is an ODE of polynomial type, then we could try to construct exact analytical solutions using the above mentioned methods.

So in our recent papers [28, 29], we used the (G'/G) -expansion method to find exact analytical solutions for geodesics in some spherically symmetric space-times of general relativity. In this paper, we use the $\exp(-\Phi(\xi))$ -expansion method to obtain two classes of exact analytical solutions of null geodesic equations in the Ellis-Bronnikov wormhole metric. Once again, this study will allow us to be convinced of the efficiency of the methods under consideration when applied to some problems of General Relativity.

2. Exponential The $\exp(-\Phi(\xi))$ - expansion method

In this section, we present the brief descriptions of $\exp(-\Phi(\xi))$ -expansion method. For this aim, we have to consider the nonlinear evolution equations (NLEE) as follows:

$$F(u,u_t,u_x;u_{xx},u_{tt},u_{tx},...)=0, \tag{1}$$

where F is a function of $u,u_t,u_x;u_{xx},u_{tt},u_{tx},...,$ and the subscripts denote the partial derivatives $u(x,t)$ of with respect to x and t .

Suppose $u(x,t) = u(\xi),\ \xi = x - Vt$, where the constant V is the velocity of the traveling wave, then the Eq. (1) reduces to a nonlinear ordinary differential equation (ODE) of the polynomial form for $u(x,t) = u(\xi)$:

$$Q(u, u', u'', u''', ...) = 0, \quad (2)$$

where Q is a function of $u, u', u'', u''', ...,$ and its derivatives point out the ordinary derivatives with respect to ξ .

In $\exp(-\Phi(\xi))$ -expansion method for solving the nonlinear ordinary differential equation (2), one consider the traveling wave solution in the form:

$$u(\xi) = \sum_{k=0}^{N} C_k \exp(-k\Phi(\xi)), \quad (3)$$

where $C_N \neq 0$, and the coefficients C_k for $0 \le k \le N$ are constants to be evaluated and $u = u(\xi)$ satisfies the first order nonlinear ordinary differential equation:

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + \mu \exp(\Phi(\xi)) + \lambda, \quad (4)$$

where μ and λ are arbitrary constants. The value of the positive integer N can be determined by balancing the highest order derivatives with the nonlinear terms of the highest order appearing in Eq. (2).

By substituting (3) into (2) and using (4) when required, we obtain a system of algebraic equations for $C_k (0 \le k \le N),\ \mu$ and λ . With the help of symbolic computation, such as, Maple, we can solve this system of algebraic equations. It is notable that Eq. (4) has the following five types of general solutions [Abdelrahman]:

$$\Phi(\xi) = \ln\left(\frac{-\sqrt{D}\tanh\left[\frac{\sqrt{D}}{2}(\xi + \xi_0)\right] - \lambda}{2\mu} \right), \quad (5)$$

when $D = \lambda^2 - 4\mu > 0,\ \mu \neq 0$,

$$\Phi(\xi)=\ln\left(\frac{\sqrt{-D}\tan\left[\frac{\sqrt{-D}}{2}(\xi+\xi_0)\right]-\lambda}{2\mu}\right), \tag{6}$$

when $D=\lambda^2-4\mu<0,\ \mu\neq 0$,

$$\Phi(\xi)=-\ln\left(\frac{\lambda}{\exp[\lambda(\xi+\xi_0)]-1}\right), \tag{7}$$

when $D=\lambda^2-4\mu>0,\ \mu=0,\ \lambda\neq 0$,

$$\Phi(\xi)=\ln\left(-\frac{2[\lambda(\xi+\xi_0)+2]}{\lambda^2(\xi+\xi_0)}\right), \tag{8}$$

when $D=\lambda^2-4\mu=0,\ \mu\neq 0,\ \lambda\neq 0$, and

$$\Phi(\xi)=\ln(\xi+\xi_0), \tag{9}$$

when $\mu=0,\ \lambda=0$. Here, ξ_0 is a constant of integration. Thus, the exact explicit solutions to the nonlinear ODE (2) can be obtained by means of the Eqs. (3) and (5)-(9).

3. Null-geodesic equation in the Ellis-Bronnikov wormhole

Here, we briefly recall the main equation of isotropic geodesic motion in the Ellis-Bronnikov wormhole spacetime, following Ref. [30] . The line element of this spacetime is described in terms of the proper radial distance r , covering the entire spacetime, by

$$ds^2=-dt^2+dr^2+(r^2+a^2)(d\theta^2+\sin^2\theta d\varphi^2), \tag{10}$$

where the coordinates are defined in the following ranges: $-\infty<t<\infty$, $-\infty<r<\infty$, $0\leq\theta\leq\pi$, and $0\leq\varphi<2\pi$. A positive parameter a defines the size of the throat of the wormhole. There are two Killing vectors in the static and spherically symmetric spacetime (10): the time translational $t^\alpha\partial_\alpha=t^t$ and axial $\varphi^\alpha\partial_\alpha=\varphi^\varphi$. It could be noted that the wormhole throat is located at $r=0$.

Due to the spherical symmetry of (10), we can put $\theta=\pi/2$ without loss of generality. The condition that the trajectory is the null geodesic can be expressed as

$$k^{\mu}k_{\mu}=0, \tag{11}$$

where k^{μ} is the wave vector of photon. Substituting the two constants of motion, the energy $E\equiv -g_{\mu\nu}t^{\mu}k^{\nu}$ and the angular momentum $L\equiv g_{\mu\nu}\phi^{\mu}k^{\nu}$ of the photon, into equation (10), we obtain the following integral of the null geodesics motion [30] :

$$\frac{1}{(r^2+a^2)^2}\left(\frac{dr}{d\varphi}\right)^2=\frac{1}{b^2}-\frac{1}{r^2+a^2}, \tag{12}$$

where $b\equiv L/E$ is the impact parameter. Obviously, $L\geq 0$ and thus $b\geq 0$. As known, the photon scatterers when the impact parameter $b>a$, but it falls into the throat at $b<a$.

One could note that the trajectories of test particles as well as the photons are usually described by the geodesics lines, i.e. $r=r(\varphi)$. At the same time, it is established that the coordinate $u(\varphi)\equiv 1/r(\varphi)$ is more convenient than $r(\varphi)$ for solving the geodesic equations in this and similar cases. Therefore, the equation (12) should be rewritten as follows

$$\left(\frac{du}{d\varphi}\right)^2=\frac{1}{b^2}\left(1+a^2u^2\right)^2-u^2-a^2u^4. \tag{13}$$

This equation can be integrated directly, but the result is expressed in elliptic functions, which is quite difficult for further analysis.

Finally, differentiating equation (13) with respect to φ , on can obtain the geodesic equation as the second-order ODE of the following form

$$\frac{d^2u}{d\varphi^2}=Au+Bu^3, \tag{14}$$

where

$$A=\frac{2a^2-b^2}{b^2},\quad B=\frac{2a^2(a^2-b^2)}{b^2}. \tag{15}$$

Now our goal is to construct an exact solution of equation (14) with the help of $\exp(-\Phi(\xi))$ -expansion method.

4. Solution of the null-geodesic equation using $\exp(-\Phi(\xi))$ -expansion method

First of all, we have to assume that the solution of ODE (14) can be presented as a polynomial in $\exp(-k\Phi(\varphi))$, i.e.

$$u(\varphi) = C_N \exp(-N\Phi(\varphi)) + ..., \tag{16}$$

where $\Phi(\varphi)$ must satisfy the auxiliary linear ODE (4) with $\xi \equiv \varphi$. By using (16) and (4), where a prime stands for the derivative with respect to φ, it can be derived that

$$u'(\varphi) = -N\, C_N \exp(-(N+1)\Phi(\varphi)) + \ldots,$$

$$u''(\varphi) = N(N+1)C_N \exp(-(N+2)\Phi(\varphi)) + \ldots, \tag{17}$$

$$u^3(\varphi) = C_N^3 \exp(-3N\Phi(\varphi)) + \ldots.$$

Using the homogeneous balance between u'' and u^3 in equation (5), followed from (16) and (17), one can get $N+2=3N$, that is $N=1$. Therefore, it is possible to represent (16) as follows

$$u(\varphi) = C_0 + C_1 \exp(-\Phi(\varphi)), \tag{18}$$

and, thus,

$$u' = -C_1 \exp(-2\Phi(\varphi)) - \lambda C$$

$$u'' = 2C_1 \exp(-3\Phi(\varphi)) + 3\lambda C_1 \exp(-2\Phi(\varphi)) + (2\mu C_1 + \lambda^2 C_1)\exp(-\Phi(\varphi)) + \lambda\mu C_1, \tag{19}$$

$$u^3 = C_0^3 + 3C_0^2 C_1 \exp(-\Phi(\varphi)) + 3C_0 C_1^2 \exp(-2\Phi(\varphi)) + C_1^3 \exp(-3\Phi(\varphi)).$$

Following the procedure of the method and substituting (18) and (19) into equation (14), we collect all terms with the same power of $\exp(-k\Phi(\varphi))$ together in the left -hand side of equation (14). As a result, this equation is converted into another polynomial in $\exp(-\Phi(\varphi))$, which coefficients should be equal to zero, that yields the following set of algebraic equations for C_0, C_1, λ and μ:

$$k=3: \quad 2C_1 - BC_1^3 = 0, \tag{20}$$

$$k=2: \quad 3\lambda C_1 - 3BC_0 C_1^2 = 0, \tag{21}$$

$$k=1: \quad 2\mu C_1 + \lambda^2 C_1 - AC_1 - 3BC_0^2 C_1 = 0, \tag{22}$$

$$k=0: \quad \lambda\mu C_1 - AC_0 - BC_0^3 = 0. \tag{23}$$

As $C_1 \neq 0$, it follows from equation (20) that $B = 2/C_1^2$ and, therefore, $B > 0$ or $b < a$, according to (15). The latter means that the following consideration is limited only to the case of a photon falling into the throat of wormhole. In addition, from the same equation (15), it follows that $A > 0$. Solving the set of equations (20)-(23), we can obtain the following set of parameters in exact solution (18): $C_0 = C$ is an arbitrary real parameter, and

$$C_1 = \sqrt{\frac{2}{B}}, \quad \mu = \frac{BC^2 + A}{2}, \quad \lambda = C\sqrt{2B} \quad \Rightarrow \quad D \equiv \lambda^2 - 4\mu = -2A < 0. \tag{24}$$

Substituting C_0 and C_1 from the latter into (18), we get

$$u(\varphi) = C + \sqrt{\frac{2}{B}} \exp(-\Phi(\varphi)), \tag{25}$$

where $\Phi(\varphi)$ can be represented as

$$\Phi(\varphi) = \ln\left(\frac{\sqrt{2A}\tan\left[\sqrt{\frac{A}{2}}(\varphi + \varphi_0)\right] - C\sqrt{2B}}{BC^2 + A}\right), \tag{26}$$

according to $D = -2A < 0$ and equation (6). Substituting this expression into Eq. (25), one can obtain the following exact solution to Eq.(14)

$$u(\varphi) = \sqrt{\frac{A}{B}} \cdot \frac{C\sqrt{B}\tan\left[\sqrt{\frac{A}{2}}(\varphi + \varphi_0)\right] + \sqrt{A}}{\sqrt{A}\tan\left[\sqrt{\frac{A}{2}}(\varphi + \varphi_0)\right] - C\sqrt{B}}, \tag{27}$$

which can be converted by a simple algebraic manipulation and some redefinition of φ_0 into the following expression

$$u(\varphi) = \sqrt{\frac{A}{B}} \tan\left[\sqrt{\frac{A}{2}}(\varphi - \varphi_0)\right] \tag{28}$$

with an arbitrary constant φ_0. Taking into account Eq. (15), the trajectories $r = r(\varphi)$, defined by this solution, are as follows

$$r(\varphi) \equiv \frac{1}{u(\varphi)} = a\sqrt{\frac{2(a^2 - b^2)}{2a^2 - b^2}} \cot\left[\sqrt{\frac{2a^2 - b^2}{2b^2}}(\varphi - \varphi_0)\right]. \tag{29}$$

Thus, a one-parameter set of photon trajectories $r = r(\varphi, \varphi_0)$ through the throat of Ellis-Bronnikov wormhole is described by equation (29) and can be illustrated by graphs in Figure 1.

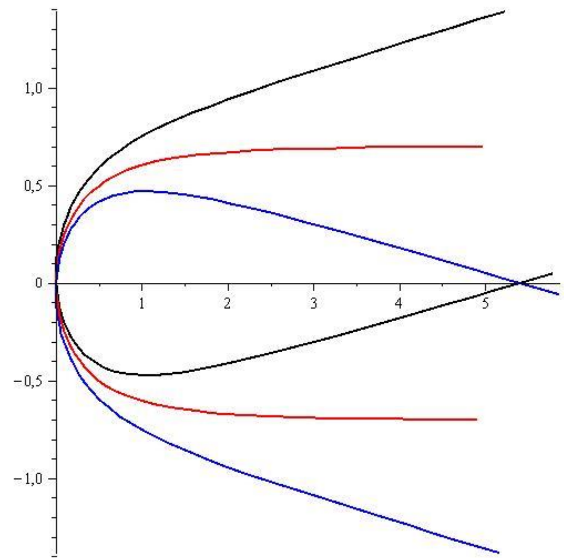

Fig. 1: Light beam trajectories $r = r(\varphi, \varphi_0)$ described by equation (29) with $a = 1, b = \sqrt{2/3}$ in $\varphi_0 = 0.13$ (black), $\varphi_0 = 0$ (red), $\varphi_0 = -0.13$ (blue).

5. Solving the null geodesics for the spherical radius

As well known, the stationary line element for the general spherically symmetric spacetime can be represented by [1]

$$ds^2 = -f(\rho)dt^2 + \frac{d\rho^2}{h(\rho)} + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{30}$$

where ρ stands for the curvature coordinate or the spherical radius.

Because of spherical symmetry, one can adopt $\theta = \pi/2$ without loss of generality, so that the second-order null-geodesic equation get the following form [30, 31]

$$\frac{d^2u}{d\varphi^2} = \frac{E^2}{2L^2}\frac{d}{du}\left[\frac{h(u)}{f(u)}\right] - h(u)u - \frac{u^2}{2}\frac{dh(u)}{du}, \tag{31}$$

where $u(\varphi) \equiv 1/\rho(\varphi)$.

In this section, we consider the metric of Ellis-Bronnikov wormhole spacetime (1) in terms of spherical radius $\rho = \pm\sqrt{r^2 + a^2}$, so that the line element becomes

$$ds^2 = -dt^2 + \frac{d\rho^2}{1 - \frac{a^2}{\rho^2}} + \rho^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{32}$$

From equations (30) and (32), we get $f(u) = 1$ and $h(u) = 1 - a^2u^2$. Substituting these functions along with $L/E = b$ into Eq.(31), we obtain the following equation for the light trajectory on the equatorial plane of the standard metric (32):

$$\frac{d^2u}{d\varphi^2} = -\left(1 + \frac{a^2}{b^2}\right)u + 2a^2u^3. \tag{33}$$

We can see that equation (33) can be represented in the form similar to (14) as

$$\frac{d^2u}{d\varphi^2} = -Au + Bu^3, \tag{34}$$

where

$$A = \frac{a^2 + b^2}{b^2} > 0, \quad B = 2a^2 > 0 \tag{35}$$

in the present case.

Comparing equations (14) and (34), we find a difference only in the sign of A. Therefore, we can repeat equations (18)-(23), changing the sign of A in them to the opposite one. As a result, an analogue of equation (8) now can be written in the following form:

$$C_1 = \sqrt{\frac{2}{B}}, \quad \mu = \frac{BC^2 - A}{2}, \quad \lambda = C\sqrt{2B}\ , \tag{36}$$

where $C_0 = C$ is an arbitrary real parameter.

As according to the latter $D = \lambda^2 - 4\mu = 2A > 0$, we get from Eq. (6) that

$$\Phi(\varphi) = \ln\left(\frac{\sqrt{2A}\tanh\left[\sqrt{\frac{A}{2}}(\varphi+\varphi_0)\right] + C\sqrt{2B}}{A - BC^2} \right). \quad (37)$$

Substituting (37) into Eq. (18), one can obtain the following exact solution to Eq.(34):

$$u(\varphi) = \sqrt{\frac{A}{B}} \cdot \frac{C\sqrt{B}\tanh\left[\sqrt{\frac{A}{2}}(\varphi+\varphi_0)\right] + \sqrt{A}}{\sqrt{A}\tanh\left[\sqrt{\frac{A}{2}}(\varphi+\varphi_0)\right] + C\sqrt{B}}. \quad (38)$$

We can introduce a new arbitrary constant C equal to a certain combination of constants, such as $\frac{\sqrt{A}+C\sqrt{B}}{\sqrt{A}-C\sqrt{B}}\exp(\sqrt{2A}\varphi_0)$. After that, the solution (38) can be represented as follows:

$$u(\varphi) \equiv \frac{1}{\rho(\varphi)} = \sqrt{\frac{A}{B}} \cdot \frac{C\exp(\sqrt{2A}\,\varphi)+1}{C\exp(\sqrt{2A}\,\varphi)-1}. \quad (39)$$

A one-parameter set of the light ray trajectories $\rho = \rho(\varphi)$ in terms of spherical radial coordinate near the Ellis-Bronnikov wormhole is described by equation (39) and can be illustrated by graphs in Figure 2.

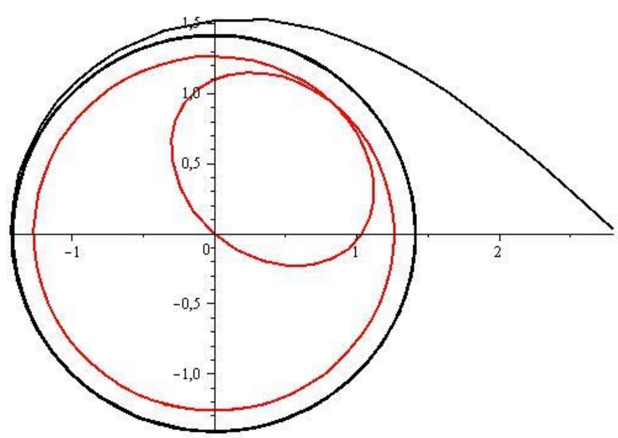

Fig. 2: Example of an image Light beam trajectories $\rho = \rho(\varphi)$ described by equation (39) with $A = 1, B = 2, C = -3$ (black), and $A = 5, B = 8, C = 10$ (red).

6. Conclusions

In this work, we used the $\exp(-\Phi(\xi))$ -expansion method to obtain two classes of the one-parameter exact analytical solutions of null geodesic equations in the Ellis-Bronnikov wormhole metric. Both of these solutions are graphically illustrated in Figure 1 and Figure 2 for a certain choice of numerical parameters in these solutions. Moreover, the resulting solutions, equations (90) and (39), coincide with similar solutions that we obtained in previous work [28] using a different method, namely the (G'/G) -expansion method.

We can conclude that application of these methods or some similar methods, mentioned in Introduction, in the manner described in this paper can make it possible to obtain some classes of exact solutions for nonlinear ODE expressed in terms of elementary functions, which greatly simplifies the analysis of the solutions in subsequent studies.

References

[1] S. Weinberg, Gravitation and Cosmology: Principles and Applications of The General Theory of Relativity (John Wiley. Press, New York, 1972).

[2] M. Visser, L. Wormholes, From Einstein to Hawking (American Institute of Physics, New York, 1996).

[3] H. G. Ellis, Ether flow through a drainhole: A particle model in general relativity, J. Math. Phys. 14, 104 (1973). https://doi.org/10.1063/1.1666161

[4] K. A. Bronnikov,Scalar-tensor theory and scalar charge, Acta Phys. Pol. B 4, 251 (1973).

[5] M. S. Morris and K. S. Thorne, Wormholes in Spacetime and Their Use for Interstellar Travel: A Tool for Teaching General Relativity, Am. J. Phys. 56, 395 (1988). https://doi.org/10.1119/1.15620

[6] M. S. Morris, K. S. Thorne and U. Yurtsever, Wormholes, Time Machines, and the Weak Energy Condition, Phys. Rev. Lett. 61, 1446 (1988). https://doi.org/10.1103/PhysRevLett.61.1446

[7] I. D. Novikov, N. S. Kardashev, A. A. Shatskii, The multicomponent Universe and the astrophysics of wormholes, UFN, Vol.177, Number 9, 1017 (2007). https://doi.org/10.3367/UFNr.0177.200709g.1017

[8] K. A. Bronnikov, L. N. Lipatova, I. D. Novikov, et al., Example of a stable wormhole in general relativity, Grav, Cosmol. 19, 269 (2013). https://doi.org/10.1134/S0202289313040038

[9] P. D. Roy, S. Aneesh, S, Kar, Revisiting a family of wormholes: geometry, matter, scalar quasinormal modes and echoes. , Eur. Phys. J. C 80, 850 (2020). https://doi.org/10.1140/epjc/s10052-020-8409-5

[10] A. Mishra, S. Chakraborty, On the trajectories of null and timelike geodesics in different wormhole geometries, Eur. Phys. J. C 78, 374 (2018)https://doi.org/10.1140/epjc/s10052-018-5854-5

[11] F. Willenborg, S.Grunau, B. Kleihaus, and J. Kunz, Geodesic motion around traversable wormholes supported by a massless conformally coupled scalar field, Phys. Rev. D 97, 124002 (2018). https://doi.org/10.1103/PhysRevD.97.124002

[12] I. Potashov, J. Tchemarina and A. Tsirulev, Null and Timelike Geodesics near the Throats of Phantom Scalar Field Wormholes, Universe 6(10), 183 (2020). https://doi.org/10.3390/universe6100183

[13] B. Ghosh, S. Dutta, S. Mukerji, S. Chakraborty, Static spherically symmetric wormhole with quadratic shape function and particle trajectories around it, Int.J.Mod.Phys.A, 36(06), 2150046 (2021). https://doi.org/10.1142/S0217751X21500469

[14] G. Panotopoulos, A. Rincon , and I. Lopes, Orbits of light rays in scale-dependent gravity: Exact analytical solutions to the null geodesic equations, Phys. Rev. D 103, 104040 (2021). https://doi.org/10.1103/PhysRevD.103.104040

[15] V. Sharma, S. Ghosh, Geodesics in Generalised Ellis-Bronnikov Spacetime Embedded in Warped 5D Background, Eur. Phys. J. C 82, 702 (2022). https://doi.org/10.1140/epjc/s10052-022-10682-6.

[16] M. Wang, Y. Zhou, Z. Li, Application of a Homogeneous Balance Method to Exact Solutions of Nonlinear Equations in Mathematical Physics, Phys. Lett. A 216(1-5), 67-75 (1996). https://doi.org/10.1016/0375-9601(96)00283-6

[17] E. Fan, H. Zhang, A note on the homogeneous balance method, Phys. Lett. A 246(5), 403-406 (1998). https://doi.org/10.1016/S0375-9601(98)00547-7

[18] A. M. Wazwaz, The tanh and the sine-cosine methods for a reliable treatment of the modified equal width equation and its variants, Communications in Nonlinear Science and Numerical Simulation, 11(2), 148-160 (2006). https://doi.org/10.1016/j.cnsns.2004.07.001

[19] E. Yusufoglu, A. Bekir, Solitons and periodic solutions of coupled nonlinear evolution equations by using the sine-cosine method, Int. J. Comp. Math. 83(12), 915-924 (2006).

[20] M. A. E. Abdelrahman, E. H. M. Zahran, M. M. A. Khater, The $\exp(-\Phi(\xi))$ -Expansion Method and Its Application for Solving Nonlinear Evolution Equations, Int. J, Modern Nonlinear Theory and Application 4, 37 (2015). https://doi.org/10.4236/ijmnta.2015.41004

[21] M. Wang and X. Li, Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation, Chaos, Solitons Fract. 24, 1257-1268 (2005). https://doi.org/10.1016/j.chaos.2004.09.044

[22] C. Dai, J. Zhang, Jacobian elliptic function method for nonlinear differential-difference equations, Chaos, Solitons Fract. 27(4), 1042-1047 (2006). https://doi.org/10.1016/j.chaos.2005.04.071

[23] S. Liu, Z. Fu, S. Liu, Q. Zhao, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, Phys. Lett. A 289(1), 69-74 (2001). https://doi.org/10.1016/S0375-9601(01)00580-1

[24] M.H. Liu, and Y.G. Zheng, New Solutions for an Elliptic Equation Method and Its Applications in Nonlinear Evolution Equations, J. Appl. Math. and Phys. 10, 2415-2431 (2022). https://doi.org/10.4236/jamp.2022.108164

[25] M. Wang, X. Li, J. Zhang, The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, Phys. Lett. A 372, 417-423 (2008). https://doi.org/10.1016/j.physleta.2007.07.051

[26] J. Zhang, X. Wei and Y. Lu, A generalized (G'/G)-expansion method and its applications, Phys. Lett. A 372, 3653-3658 (2008). https://doi.org/10.1016/j.physleta.2008.02.027

[27] J. Zhang, F. Jiang and X. Zhao, An improved (G'/G)-expansion method for solving nonlinear evolution equations, Int. J. Comput. Math. 87(8), 1716-1725 (2010). https://doi.org/10.1080/00207160802450166

[28] V. K. Shchigolev, Exact solutions to the null-geodesics in Ellis-Bronnikov wormhole spacetime via (G'/G)-expansion method, Modern Physics Letters A, Vol. 37, No. 20, 2250124, 2022. https://doi.org/10.1142/S0217732322501243

[29] V. K. Shchigolev, Exact analytical solutions to the geodesic equations in general relativity via (G'/G) - expansion method, Gen Relativ Gravit, Vol. 54, 78. 2022. https://doi.org/10.1007/s10714-022-02964-x

[30] N. Tsukamoto, Strong deflection limit analysis and gravitational lensing of an Ellis wormhole, Phys. Rev. D 94, 124001 (2016). https://doi.org/10.1103/PhysRevD.94.124001

[31] Ya-Peng Hu, Hongsheng Zhang, Jun-Peng Hou, and Liang-Zun Tang, Perihelion Precession and Deflection of Light in the General Spherically Symmetric Spacetime, Adv.High Energy Phys. 2014, 604321 (2014). https://doi.org/10.1155/2014/604321