Isochronous Behavior and Diophantine Relations for Complex-Valued Nonlinear Systems

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(Received 16 November 2022, Accepted 24 December 2022, Published 31 December 2022)

Abstract

Isochronous systems are not rare in dynamical systems. Three complex-valued nonlinear systems (quadratic and cubic nonlinearity, van der Pol, gyroscopic oscillator) are investigated by an asymptotic perturbation method based on Fourier expansion and time rescaling. Four coupled equations for the amplitude and the phase of solutions are derived. Approximate solutions are obtained and their stability is discussed. We find that in the first two cases the motion is periodic, while in the third case the motion is periodic only if appropriate Diophantine relations are satisfied. Analytic approximate solutions are checked by numerical integration.

Keywords: : periodic motion, complex-valued system, asymptotic analysis, Diophantine equations.

1. Introduction

In the last years researchers began to consider extensively isochronous behavior in nonlinear oscillators but above all in autonomous systems [1-3]. A system is called isochronous when it shows in its phase space a sector where all its solutions are periodic. We want to extend this investigation about isochronous system for the complex-valued systems. complex-valued nonlinear dynamical systems have been extensively studied, by means of approximate analytical and numerical methods . The complex-valued nonlinear differential equations appears in many fields of science, for instance rotor dynamics, high-energy particle accelerators, robots and shells[4]. Helleman [5] and Bountis and Mahmoud [6] have carefully studied the existence and stability of periodic orbits for a complex-valued nonlinear system describing colliding beams. Mahmoud and Aly [7] used the indicatrix method to detect the existence of periodic orbits of the same nonlinear system. By the generalized averaging method they obtained approximate analytical solutions for periodic orbits with period equal to the damping force period (phase-locked solutions) and investigated their stability. Manasevich *et al.* [8] investigated periodic solutions of some complex-valued Liénard and Rayleigh equations. Mamoud [9] obtained approximate solution of a class of complex nonlinear dynamical systems. In a series of papers [10-13] Cveticanin developed a method for solving complex-valued nonlinear systems. Maccari [14] investigated period and quasi-periodic solutions of a complex-valued nonlinear system.

In this paper we study the complex-valued quadratic and cubic nonlinear oscillator,

$$\ddot{z} + \omega^2 z + a_1 z^2 + a_2 z^3 = 0, \tag{1}$$

the complex-valued van der Pol oscillator,

$$\ddot{z} + \omega^2 z + a_4 \dot{z} + a_6 z^2 \dot{z} = 0, \tag{2}$$

and finally the gyroscopic function in the following form

$$\ddot{z} + \omega^2 z + a_3 |\dot{z}|^2 z + i a_5 \dot{z} + + i a_7 |z|^2 \dot{z} = 0, \tag{3}$$

where $z(t) = x(t) + \mathrm{i}y(t)$, the dot denotes differentiation with respect to the time and $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ are appropriate parameters. The paper is arranged as follows.

In Section 2 we use a suitable perturbation method [15-16], calculate the lowest order approximate analytic solution of the equation (1) and derive a non-linear system of four coupled differential equations in the phase and amplitude of solutions. We demonstrate that the phase-locked solutions and isochronous systems corresponding to bounded motion are possible for this nonlinear system.

In Section 3 the same analysis is performed for the complex-valued van der Pol oscillator and one again isochronous solutions are possible and the corresponding steady-state finite amplitude solutions are derived.

At last Section 4 we study the gyroscopic function and show that periodic solutions are possible only if appropriate Diophantine relations are verified

Analytic approximate solutions are constructed and compared with numerical integration.

Final considerations are exposed in Section 5.

2. The complex-valued quadratic and cubic nonlinear oscillator

From the equation (1) we see that the complex-valued nonlinear system with quadratic and cubic nonlinearities corresponds to two coupled nonlinear oscillators:

$$\ddot{X} + \omega^2 X + a_1(X^2 - Y^2) + a_2 X(X^2 - 3Y^2) = 0, \tag{4}$$

$$\ddot{Y} + \omega^2 Y + 2a_2 XY + a_3 Y(3X^2 - Y^2) = 0. \tag{5}$$

We now introduce the slow time

$$\tau = \varepsilon^2 t, \tag{6}$$

and with the substitution $a_1 \to \varepsilon a_1$, $a_2 \to \varepsilon^2 a_2$, equation (1) yields

$$\ddot{z} + \omega^2 z + \varepsilon a_1 z^2 + \varepsilon^2 a_2 z^3 = 0 \tag{7}$$

We assume for the equation (7) a solution *z(t)* of the form

$$z(t) = \sum_{n=-\infty[\mathrm{odd}]}^{+\infty} \varepsilon^{\gamma_n} \psi_n(\tau, |\varepsilon) \exp(-\mathrm{in}\omega t), \tag{8}$$

i.e. a power series in the expansion parameter □ (a bookkeeping device that will be set equal to unity in the final analysis), with

$$\gamma_n = |n| - 1. \tag{9}$$

For the system analysis not only time rescaling but also Fourier expansion are needed, because the complex oscillator reduces for a vanishing value of the parameter e to a simple harmonic oscillator. For small values of e, we observe a slow modulation of the coefficients of the Fourier expansion.

The assumed solution (8) can be written more explicitly

$$z(t) = \Big(\psi_1 \exp(-i\omega t) + \varepsilon\psi_2 \exp(-2i\omega t) + \psi_{-1}\exp(i\omega t) + \varepsilon\psi_{-2}\exp(2i\omega t)\Big) + \psi_0 + O(\varepsilon^2), \tag{10}$$

and we see that it can be considered a combination of the various harmonics with coefficients depending on □ and □. In the following, and in order to simplify our calculations, we will use the notations

$$\psi_1 = \psi \quad \psi_{-1} = \phi \ . \tag{11}$$

Note that the introduction of the slow time (6) implies that

$$\frac{d}{\mathrm{dt}} \to \left(\frac{d}{d\tau} - in\omega\right). \tag{12}$$

Using equation (8) and substituting into equation (7) yields various equations for each harmonic *n* and for a fixed order of approximation on the perturbation parameter ε.

For $n=\pm 1$, we can derive two differential equations for the evolution of the complex amplitudes ψ and ϕ,

$$i\frac{d\psi}{d\tau} = \alpha_1 \psi^2 \phi, \tag{13}$$

$$i\frac{d\phi}{d\tau} = -\alpha_1 \psi \phi^2. \tag{14}$$

where

$$\alpha_1 = \left(\frac{3a_2}{2\omega} - \frac{5a_1^2}{3\omega^3}\right) \ . \tag{15}$$

Substituting the polar forms,

$$\psi(\tau) = \rho(\tau)\exp\big(i\vartheta(\tau)\big), \phi(\tau) = \chi(\tau)\exp\big(i\alpha(\tau)\big), \tag{16}$$

into equations (13-14), and separating real and imaginary parts, we arrive at the following nonlinear system

$$\frac{d\rho}{d\tau} = \alpha_1 \rho^2 \chi \sin(\vartheta + \alpha), \tag{17}$$

$$\frac{d\chi}{d\tau} = -\alpha_1 \rho \chi^2 \sin(\vartheta + \alpha), \tag{18}$$

$$\frac{d\vartheta}{d\tau} = -\alpha_1 \rho\chi \cos(\vartheta + \alpha), \tag{19}$$

$$\frac{d\alpha}{d\tau} = \alpha_1 \rho\chi \cos(\vartheta + \alpha). \tag{20}$$

From (19-20) we see that

$$\vartheta + \alpha = cost \tag{21}$$

From (17-18) we easily get

$$\rho\chi = \rho_0\chi_0 = cost, \quad . \tag{22}$$

where ρ_0 and χ_0 are the initial conditions, $\rho_0 = \rho(0), \chi_0 = \chi(0)$. Phase-locked periodic solutions of the complex-valued nonlinear system (7) $(d\rho/d\tau = d\chi/d\tau = d\vartheta/d\tau = d\alpha/d\tau = 0)$ are impossible. However, steady-state finite-amplitude solutions of the equations exist and are given by

$$\alpha + \vartheta = 0 \text{ or } \pi, \qquad \rho = \rho_0 = const, \qquad \chi = \chi_0 = const\ . \tag{23}$$

The first of the conditions (23) is also requested to eliminate unbounded solutions.

The first order bounded approximate solutions are $(\alpha + \vartheta = 0)$

$$X = (\rho_0 + \chi_0)\cos\big((\alpha_1\rho_0\chi_0 + \omega)t\big), \tag{24}$$

$$Y = (\chi_0 - \rho_0)\sin\big((\alpha_1\rho_0\chi_0 + \omega)t\big), \tag{25}$$

and $(\alpha + \vartheta = \pi)$

$$X = (\rho_0 - \chi_0)\cos\big((\alpha_1\rho_0\chi_0 + \omega)t\big), \tag{26}$$

$$Y = -(\rho_0 + \chi_0)\sin\big((\alpha_1\rho_0\chi_0 + \omega)t\big), \tag{27}$$

where

$$\Omega = \omega + \alpha_1\rho_0\chi_0, \tag{28}$$

or respectively

$$z(t) = \rho_0\exp(-i\Omega t) + \chi_0\exp(i\Omega t), \tag{29a}$$

$$z(t) = \rho_0\exp(-i\Omega t) + \chi_0\exp(i\Omega t). \tag{29b}$$

In Fig. 1 we show a comparison between the approximate solution (24-25) and the numerical solution of the same motion. We represent a projection of the associated map of the equation (1), obtained with the values *(X(0),Y(0)), (X(T),Y(T)), (X(2T),Y(2T))*, ..., where *T* is the period,

$$T = \frac{2\pi}{\Omega}. \tag{30}$$

Crosses represent the approximate solution and boxes represent the numerical solution. The characteristic closed curve reveals that the motion is periodic ($\rho_0 = 0.8$, $\chi_0 = 0.4$, $\omega = 1$, $a_1 = a_2 = 0.1$).

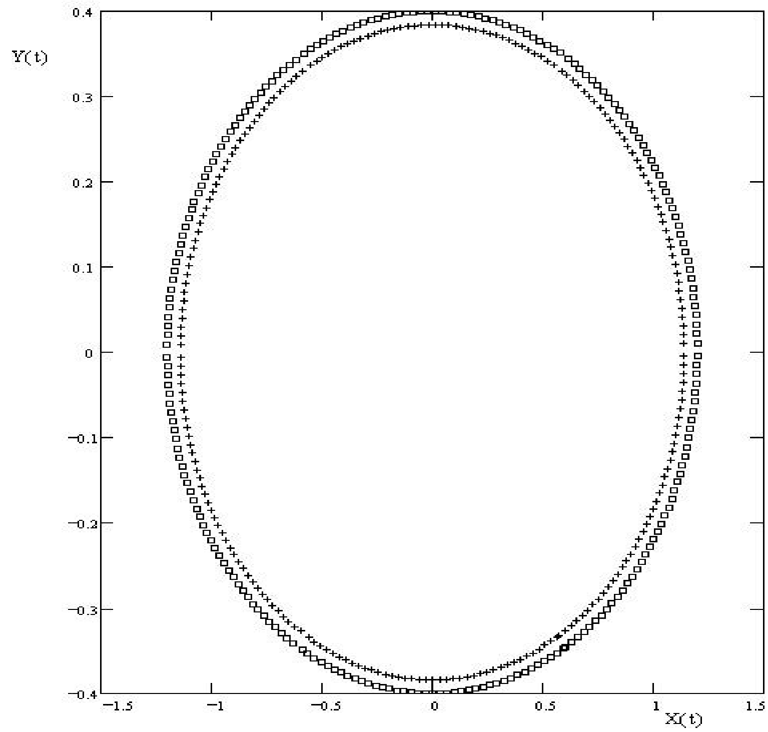

FIGURE 1: Ccomparison between the approximate solution (24-25) and the numerical solution of the same motion. We represent a projection of the associated map of the equation (1), obtained with the values *(X(0),Y(0)), (X(T),Y(T)), (X(2T),Y(2T))*, ..., where T is the period, Crosses represent the approximate solution and boxes represent the numerical solution. The characteristic closed curve reveals that the motion is periodic ($\rho_0 = 0.8$, $\chi_0 = 0.4$, $\omega = 1$, $a_1 = a_2 = 0.1$).

The validity of the approximate solution should be expected to be restricted on bounded intervals of the t-variable and then on time-scale $t = O\left(\frac{1}{\varepsilon^2}\right)$. If one wishes to construct approximate solutions on larger intervals such that $t = O\left(\frac{1}{e}\right)$ then the higher terms will in general affect the solution and must be included (see Section 2). Moreover, the approximate solution (24-25) and (26-27) will be within $O(e)$ of the true solution on bounded intervals of the$t$-variable, and, if the solution is periodic, for all *t*.

3. The complex-valued van der Pol oscillator

The complex-valued van der Pol oscillator correspond to two coupled nonlinear differential equations:

$$\ddot{X} + \omega^2 X + a_4 \dot{X} + a_6 \left((X^2 - Y^2)\dot{X} - 2XY\dot{Y}\right) = 0 \tag{31}$$

$$\ddot{Y} + \omega^2 Y + a_4 \dot{Y} + a_6 \left((X^2 - Y^2)\dot{Y} + 2XY\dot{X} \right) = 0 \tag{32}$$

With the substitution $a_4 \to \varepsilon^2 a_4$, $a_6 \to \varepsilon^2 a_6$, equation (2) yields the nonlinear system

$$\frac{d\psi}{d\tau} = -\frac{1}{2} a_4 \psi - \frac{1}{2} a_6 \psi^2 \phi, \tag{33}$$

$$\frac{d\phi}{d\tau} = -\frac{1}{2} a_4 \phi - \frac{1}{2} a_6 \psi \phi^2. \tag{34}$$

or with the substitution (16)

$$\rho_\tau = -\frac{a_4}{2} \rho - \frac{a_6}{2} \rho^2 \chi \cos(\vartheta + \alpha), \tag{35}$$

$$\chi_\tau = -\frac{a_4}{2} \chi - \frac{a_6}{2} \rho \chi^2 \cos(\vartheta + \alpha), \tag{36}$$

$$\vartheta_\tau = -\frac{a_6}{2} \rho \chi \sin(\vartheta + \phi), \tag{37}$$

$$\alpha_\tau = -\frac{a_6}{2} \rho \chi \sin(\vartheta + \phi), \tag{38}$$

We can see from equations (35-36) that

$$\frac{\rho}{\chi} = \frac{\rho_0}{\chi_0} = cost\,, \tag{39}$$

where ρ_0 and χ_0 are the initial conditions.

The only acceptable solution for steady-state finite amplitude solutions are given by ($a_4<0$ and $a_6>0$ as in the standard van der Pol oscillator)

$$\vartheta + \alpha = 0, \qquad \rho_0 \chi_0 = -\frac{a_4}{a_6}. \tag{40}$$

The stability properties of the above illustrated fixed-point solutions are examined by applying the well-known method of linearization. We superpose small perturbations in the steady state solution and the resulting equations are then linearized. Subsequently we consider the eigenvalues of the corresponding system of first order differential equations with constant coefficients (the Jacobian matrix). A positive real root indicates an unstable solution, whereas if the real parts of the eigenvalues are all negative then the steady state solution is stable.

The eigenvalues of the Jacobian matrix of the nonlinear system (35-38) are all negative for the standard choice

$$a_4 < 0, a_6 > 0. \tag{41}$$

The first order approximate periodic solution is given by

$$X(t) = \rho_0 \cos(\omega t - \vartheta_0) - \frac{a_4}{a_6 \rho_0} \cos(\omega t + \alpha_0), \tag{42a}$$

$$Y(t) = -\rho_0 \sin(\omega t - \vartheta_0) - \frac{a_4}{a_6 \rho_0} \sin(\omega t + \alpha_0)\,. \tag{42b}$$

or

$$z(t) = \rho_0 \exp\big(-i(\omega t - \vartheta_0)\big) + \chi_0 \exp\big(i(\omega t + \alpha_0)\big). \tag{43}$$

In Fig. 2 we show a comparison between the approximate solution (42) and the numerical solution of the same motion. We represent a projection of the associated map of the equation (2), obtained with the values *(X(0),Y(0)), (X(T),Y(T)), (X(2T),Y(2T))*, ..., where *T* is the period,

$$T = \frac{2\pi}{\omega}. \tag{44}$$

Crosses represent the approximate solution and boxes represent the numerical solution. The characteristic closed curve reveals that the motion is periodic ($\rho_0 = 0.5,\ \chi_0 = 0.8,\ \omega = 1,\ a_4 = -0.1, a_6 = 0.1$).

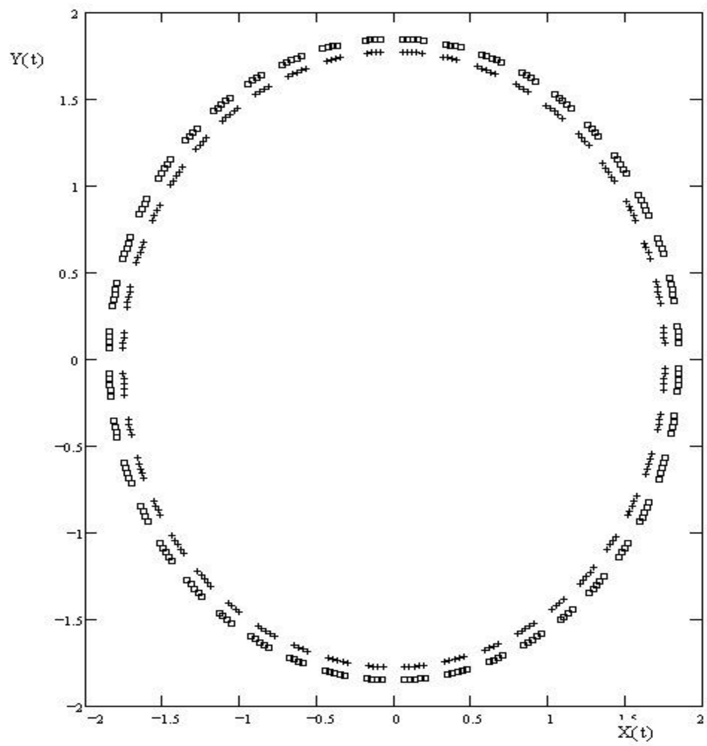

FIGURE 2: Projection on the (*X(t), Y(t)*) plane of the associated map of the complex-valued system (2). Crosses are the approximate solution and boxes the numerical solution.

4. The complex-valued gyroscopic function

The complex-valued gyroscopic function corresponds to two coupled nonlinear differential equations:

$$\ddot{X}+\omega^2X+a_3(X^2+Y^2)X-a_5\dot{Y}-a_7(X^2+Y^2)\dot{Y}=0, \tag{45}$$

$$\ddot{Y}+\omega^2Y+a_3(X^2+Y^2)Y+a_5\dot{X}-a_7(X^2+Y^2)\dot{X}=0 \tag{46}$$

With the substitution $a_3\to\varepsilon^2a_3$, $a_5\to\varepsilon^2a_5$, $a_7\to\varepsilon^2a_7$, equation (3) yields the nonlinear system

$$i\frac{d\psi}{d\tau}=\frac{a_3}{2\omega}(|\psi|^2+|\phi|^2)\psi+\frac{a_5}{2}\psi+\frac{1}{2}a_7(|\psi|^2+|\phi|^2)\psi, \tag{47}$$

$$i\frac{d\phi}{d\tau}=-\frac{a_3}{2\omega}(|\psi|^2+|\phi|^2)\phi+\frac{a_5}{2}\phi+\frac{1}{2}a_7(|\psi|^2+|\phi|^2)\phi, \tag{48}$$

or with the substitution (16)

$$\rho_\tau=0,\ \rho=\rho_0\ ,\ \chi_\tau=0,\ \chi=\chi_0\ , \tag{49}$$

$$\vartheta_\tau=-\left(\frac{a_3}{2\omega}+\frac{a_7}{2}\right)(\rho_0^2+\chi_0^2)-\frac{a_5}{2}=\Omega_1, \tag{50}$$

$$X(t)=\rho_0\cos\big((\omega-\Omega_1)t-\vartheta_0\big)+\chi_0\cos\big((\omega+\Omega_2)t+\alpha_0\big) \tag{51}$$

where ρ_0 and χ_0 are the initial conditions.

The first order approximate periodic solution is given by

$$X(t)=\rho_0\cos\big((\omega-\Omega_1)t-\vartheta_0\big)+\chi_0\cos\big((\omega+\Omega_2)t+\alpha_0\big), \tag{52}$$

$$Y(t)=-\rho_0\sin\big((\omega-\Omega_1)t-\vartheta_0\big)+\chi_0\sin\big((\omega+\Omega_2)t+\alpha_0\big)\ . \tag{53}$$

or

$$z(t)=\rho_0\exp\Big(-i\big((\omega-\Omega_1)t-\vartheta_0\big)\Big)+\chi_0\exp\Big(i\big((\omega+\Omega_2)t+\alpha_0\big)\Big). \tag{54}$$

In general we see from (54) that we get a two period quasi-periodic motion, but if the following Diophantine relation

$$\omega-\Omega_1=\frac{p}{q}(\omega+\Omega_2), \tag{55}$$

is verified (p and q are integers), then motion is simply periodic. Equation (55) with (50) and (51) yields

$$\rho_0^2+\chi_0^2=\omega\frac{p(2\omega-a_5)-q(a_5+2\omega)}{p(\omega a_7-a_3)+q(\omega a_7+a_3)} \tag{56}$$

In the plane (ρ,χ) we find infinite circles with radius given by the square root of the r.h.s.. If the initial conditions are on these circles, the corresponding solution is periodic.

We list the most simple solutions

$$p=q=1\quad \rho_0^2+\chi_0^2=-\frac{a_5}{a_7}, \tag{57}$$

$$p=2,q=1\quad \rho_0^2+\chi_0^2=\left(\frac{2\omega-3a_5}{3a_7\omega-a_3}\right)\omega, \tag{58}$$

$$p = 1, q = 2 \quad \rho_0^2 + \chi_0^2 = -\left(\frac{3a_5+2\omega}{3a_7\omega+a_3}\right)\omega, \tag{59}$$

$$p = 1, q = 3 \quad \rho_0^2 + \chi_0^2 = -2\left(\frac{a_5+\omega}{2a_7\omega+a_3}\right)\omega, \tag{60}$$

$$p = 3, q = 1 \quad \rho_0^2 + \chi_0^2 = 2\left(\frac{\omega-a_5}{2a_7\omega-a_3}\right)\omega, \tag{61}$$

$$p = 2, q = 3 \quad \rho_0^2 + \chi_0^2 = -\left(\frac{2\omega+5a_5}{5a_7\omega+a_3}\right)\omega, \tag{62}$$

$$p = 3, q = 4 \quad \rho_0^2 + \chi_0^2 = -\left(\frac{2\omega+7a_5}{7a_7\omega+a_3}\right)\omega, \tag{63}$$

$$p = 4, q = 5 \quad \rho_0^2 + \chi_0^2 = -\left(\frac{2\omega+9a_5}{9a_7\omega+a_3}\right)\omega, \tag{64}$$

For example, we consider the case (64) with $a_3=0.1$, $a_5=a_7=-0.1$, $\omega = 1$, with

$$\rho_0^2 + \chi_0^2 = 1.375,\ \rho_0 = 0.857, \chi_0 = 0.800. \tag{65}$$

In Fig. 3 we show a comparison between the approximate solution (52-53) and the numerical solution of the same motion. We represent a projection of the associated map of the equation (3), obtained with the values *(X(0),Y(0)), (X(T),Y(T)), (X(2T),Y(2T))*, ..., where *T* is the period,

$$T = \frac{2\pi}{\omega-\Omega_1}. \tag{66}$$

Crosses represent the approximate solution and boxes represent the numerical solution. The characteristic closed curve reveals that the motion is periodic,

$$\omega - \Omega_1 = \frac{4}{5}(\omega + \Omega_2). \tag{67}$$

The agreement of the results is excellent, because the maximum difference is *0.1* and the medium difference is *0.07*, i.e. of order e as expected.

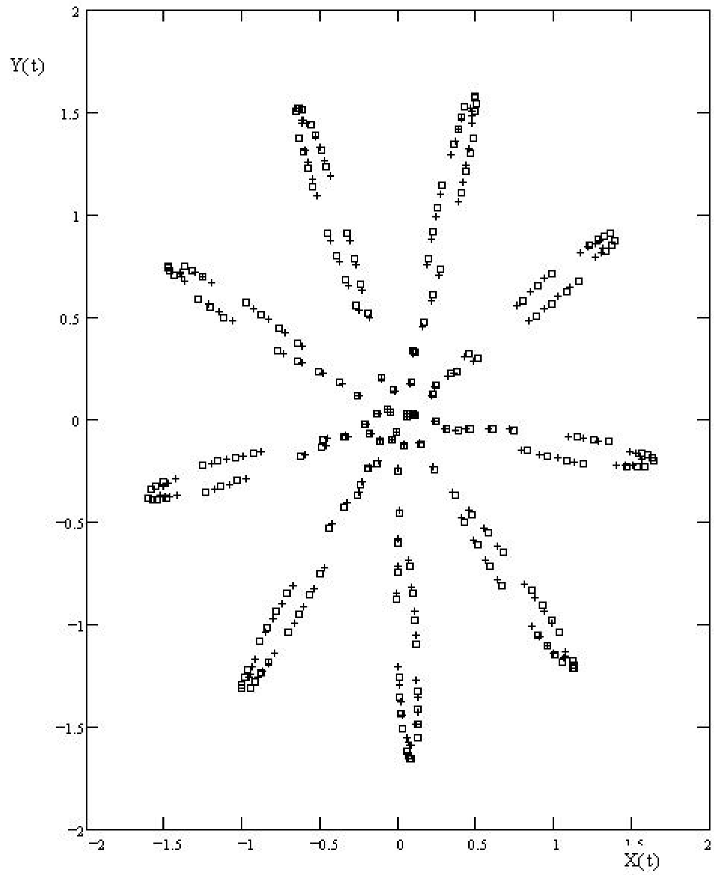

FIGURE 3: Projection on the (*X(t), Y(t)*) plane of the associated map of the complex-valued system (3). Crosses are the approximate solution and boxes the numerical solution.

In Fig.4 we show a two period quasi-periodic motion corresponding to

$$\rho_0 = 0.857, \chi_0 = 0.900 \tag{68}$$

Note that the condition (56) is not verified.

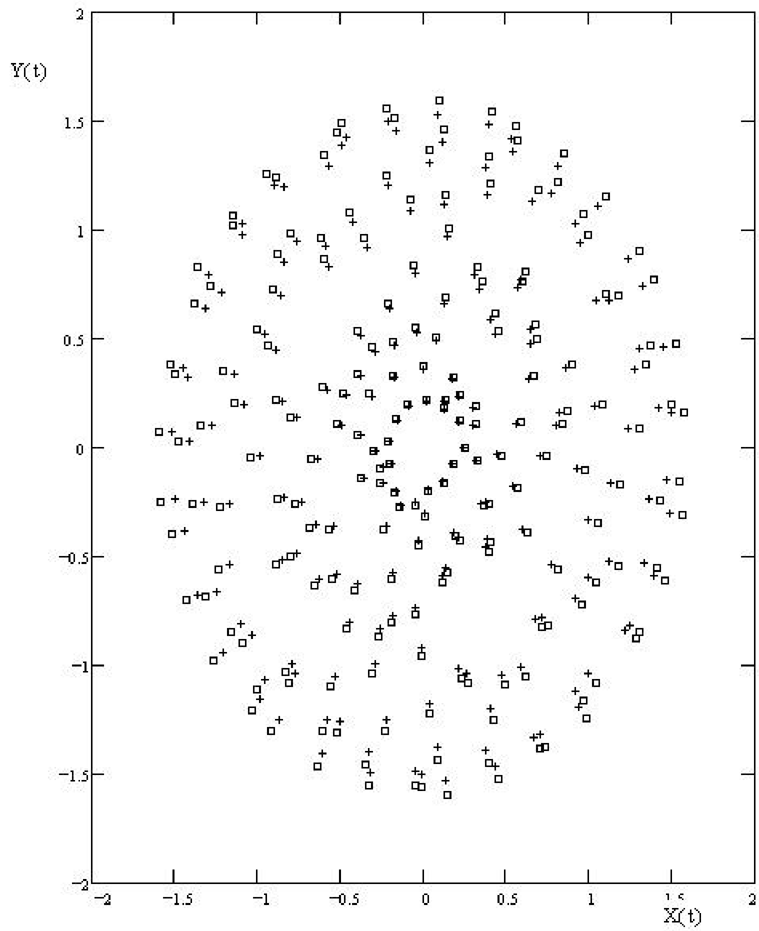

FIGURE 4: Projection on the (*X(t), Y(t)*) plane of the associated map of the complex-valued system (3). Crosses are the approximate solution and boxes the numerical solution.

5. Conclusion

Three complex-valued nonlinear systems (quadratic and cubic nonlinearity, van der Pol, gyroscopic oscillator) have been investigated by an asymptotic perturbation method based on Fourier expansion and time rescaling. Four coupled equations for the amplitude and the phase of solutions have been derived. We have demonstrated that in the first two cases the motion is periodic, while in the third case the motion is periodic only if appropriate Diophantine relations are satisfied. Analytic approximate solutions have been checked by numerical integration.

A direct extension of this work can be given by the introduction of other nonlinear terms or resonances (for example the fundamental 1:1 or the principal 1:2 resonances).

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