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Vibration Control for Liènard Systems

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Abstract

Self-excitations can be dangerous in many nonlinear systems and can produce catastrophic failures, that is a sudden and complete failure that cannot be put right. We extend the nonlocal vibration control to the suppression of the self-excited vibrations of the Liènard system. We introduce a non local control force that yields a third order non-linear differential equation and use a nonlocal active control to mitigate the amplitude peak in the self-excitations. The nonlocal parameters can be carefully adjusted, in order to avoid undesirable behavior and dynamical nonlinear excitations. We consider the effects of changing the nonlocal parameters on the stability and the value of the response of the system under control. We demonstrate that our method can successfully improve the self-excitation active control, studying a Liènard system through the (AP) asymptotic reduction method. A nonlocal force can be used to suppress self-excitations and put under control the oscillator behavior.

Keywords: : Liènard system, vibration control, active control, reduction method, self-excitations

1. Introduction

Nonlinear ordinary differential equations can describe many physical phenomena and moreover sometimes they can be symmetry reductions of nonlinear partial differential equations [1]. Self excitations are common in these systems.

Let f and g be two continuously differentiable functions on **R** with f an odd function and g a generic function. The Liènard equation is a second-order differential equation in the following form

$$\frac{d^2X}{dt^2} + g(X)\frac{dX}{dt} + f(X) = 0 \qquad (1.1)$$

For instance, The well known van der Pol oscillator is a Liènard system

Under certain additional assumptions the Liènard theorem ensures the uniqueness and existence of a limit cycle for such a system. In particular, a Liènard system has a unique and stable limit cycle surrounding the origin if it satisfies the following additional properties

(i) g(x) >0 for all x>0;

(ii) F(x) has exactly one positive root at some value ρ, where F(x)<0 for 0<x<ρ and F(x)>0 and monotonic for x>ρ,

where

$$F(x) = \int_0^x f\,(\rho)d\rho \tag{1.2}$$

and

$$\lim_{x\to\infty} F\,(x) = \infty \tag{1.3}$$

Not so many papers are devoted to this topic, vibration control for self-excited Liènard systems so we mention a few papers about similar issues. J. Warminski et al. [2] perform an active vibration control in a nonlinear beam with self- and external excitations. Using the nonlinear saturation control (NSC) algorithm and multiple time scales method they can find a first order approximate solution and compare it with numerical solution. Y. J. F. Kpomahou [3] et al. investigated the nonlinear dynamics and active control in a Liènard system under parametric and external excitations. It is found that for an appropriate choice of the gain parameter, then the chaotic behavior is completely removed. Maccari A. [4] used a traditional control method based upon time delay state feedback for a parametrically excited Liènard system in order to reduce the amplitude peak of the parametric resonance and to exclude the existence of two-period quasi-periodic motion, he finds the appropriate choices for the nonlocal parameters and the time delay. Xu and Lu studied the Hopf bifurcation of time delayed Liènard systems [5]. Yoshitake et al.[6] investigated the vibration of a forced self-excited system with time delay finding many important characteristics. Maccari investigated the fundamental [7] and the primary [8] resonance of a van der Pol system and found that the vibration control and high amplitude response suppression are possible using the state feedback control with a time delay. Belhaq and Sa [9] shown that a fast vertical parametric excitation can be used to suppress self-excitations in a delayed van der Pol oscillator.

The term jerk dynamics for the third–order differential equations in the case that the dependent variable the displacement was introduced by Schot [10]. Maccari [11, 12, 13] investigated the nonlocal oscillator, i.e. an oscillator subjected to a nonlocal force that is equivalent to a third order differential equation. He used an asymptotic perturbation techniques which combine the harmonic balance procedure and the method of multiple time scales. Linz studied the connection between one–dimensional jerk dynamics and nonlinear dynamical systems in three–dimensional phase space [14, 15]. Gottlieb [16] found periodic solutions and limit cycles [17-18] for some simple jerk equations by means of harmonic balance methods. Wu *et al.* [19] proposed an improved harmonic balance method for nonlinear jerk equations, while Ma et al. [20] and Hu [21] used a first–order harmonic balance procedure with a parameter perturbation technique

In this paper, a novel strategy for suppressing the self-excitations of Liènard systems (1.1) is investigated.

The paper is arranged as follows. In **Section 2** we consider the Liènard system (1.1) with a nonlocal vibration control term and use the asymptotic perturbation (AP) method [4] in order to obtain analytic approximate solutions. Three steps are used by the AP method, i) introducing a slow time scale, ii) obtaining the form of solution in terms of harmonic components, and iii) solving directly for the various harmonic components via harmonic balance. We can use a standard procedure to derive increasingly accurate solutions by increasing the order of approximation in terms of the small parameter □. The first-order approximate solution is identical to that obtainable with the other perturbation methods. We underline that there may be other solutions, which the slow flow equations do not describe (for example large-amplitude quasi-periodic motion or chaotic behavior,). The most useful characteristics of harmonic balance and multiple scale methods are used by means of the AP method.

Two slow-flow equations on the amplitude and the phase are obtained. Steady state solutions (corresponding to periodic motion) and their stability are discussed.

In **Section 3** the nonlocal parameters are chosen by analysing the modulation equations of the amplitude and the phase. We consider the effects of changing the nonlocal parameters on the value of the response of the system under control. We demonstrate that, from the viewpoint of vibration control, a correct choice of the nonlocal parameters can enhance the control performance and suppress the amplitude peak of the self-excited response in Liènard systems.

Finally, in the S**ection 4**, we summarise the most important results and indicate some possible extensions and generalisations.

2. The mathematical framework

In particular we study a Lienard system where

$$f(X) = f_3 X^3 \tag{2.1}$$

and

$$g(X) = -g_0 - g_1 X - g_2 X^2 \tag{2.2}$$

and study the integro-differential equation

$$\frac{d^2X}{dt^2} + g(X)\frac{dX}{dt} + f(X) = F_{NL}(t) = \int_o^t (AX + BX^2 + CX^3)\, dt' \tag{2.3}$$

where $F_{NL}(t)$is the nonlocal force and *A, B, C* appropriate control parameters. The integro-differential (2.3) is equivalent to

$$\frac{d^3X}{dt^3} + \frac{df}{dX}\frac{dX}{dt} + \frac{dg}{dX}\left(\frac{dX}{dt}\right)^2 + g(X)\frac{d^2X}{dt^2} = (AX + BX^2 + CX^3) \tag{2.4}$$

with initial conditions

$$\frac{d^2X}{dt^2}(t = t_0) + f(X_0) + g(X_0)\frac{dX}{dt}(t = t_0) = 0 \tag{2.5}$$

The nonlocal force includes the whole precedent temporal evolution of a given particle, and not only its actual position. Using the AP method we assume weak damping and linear and quadratic nonlocal parameters and scale the coefficients,

$$(f_0, A, B) \to \varepsilon^2 (f_0, A, B), \tag{2.6}$$

where ε is a small nondimensional parameter that is artificially introduced to serve as bookkeeping device and will be set equal to unity in the final analysis.

Now we can introduce the slow variable

$$\tau = \varepsilon^2 t, \tag{2.7}$$

In other words, we need to look on larger scales, to get a non negligible contribution by nonlinear and control terms.

The solution *X(t)* of equation (2.4) can be expressed by means of a power series in the expansion parameter □,

$$X(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\gamma_n} \psi_n(\xi; |\varepsilon) \exp(-\mathrm{in}\omega t), \tag{2.8}$$

where $\gamma_n = |n|$ for $n \neq 0$, $\gamma_0 = 2$ and $\psi_m(\tau, \varepsilon) = \psi\tilde{-}_m(\tau, \varepsilon)$.

Equation (2.8) can be written more explicitly

$$X(t) = \varepsilon^2 \psi_0(\tau; \varepsilon) + \varepsilon(\psi_1(\tau; \varepsilon) \exp(-i\omega t) + c.c.) + h.o.t., \tag{2.9}$$

where *h.o.t.* = higher order terms and *c.c.* stands for complex conjugate of the preceding terms. The functions $\psi_n(\tau, \varepsilon)$ depend on the parameter □ and we suppose that the limit of the ψ_n for $e \to 0$ exists and is finite and moreover they can be expanded in power series of ε, i.e.

$$\psi_n(\tau; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \, \psi_n^{(i)}(\tau). \tag{2.10}$$

In the following for simplicity we use the abbreviations $\psi_n^{(0)} = \psi_n$ for $n \neq 1$ and $\psi_0^{(0)} = \chi$ for *n*=0. In the lowest order calculations, only the functions corresponding to *i=0* appear.

The solution is then a Fourier expansion in which the coefficients vary slowly in time and the lowest order terms correspond to the harmonic solution of the linear problem. Evolution equations for the amplitudes of the harmonic terms are then derived by substituting the expression of the solution into the original equations and projecting onto each Fourier mode.

For *n=1* we obtain the linear equation (order ε^2)

$$-2\omega^2 \psi_\tau - 3i\omega f_3 |\psi|^2 \psi + g_0 \omega^2 \psi + 5 g_2 \omega^2 |\psi|^2 \psi - g_1 \omega^2 (\psi_2 \bar{\psi}) + g_1 \omega^2 \chi \psi = 0. \tag{2.11}$$

For n=2 we get

$$\psi_2 = i\gamma_2 \psi \qquad \gamma_2 = \frac{4 g_1}{3\omega} \tag{2.12}$$

Considering equation (1.1) for *n=0* yields (order ε) , we obtain

$$\omega^2 \chi_\tau = A\chi + 2B|\psi|^2, \tag{2.13}$$

We can derive a differential equation for the evolution of the complex amplitude$y$,

$$\psi_\tau = \alpha_1 \psi + (\beta_1 + i\gamma_1)|\psi|^2 \psi + \delta \chi \psi, \tag{2.14a}$$

$$\chi_\tau = \alpha \chi + \beta \rho^2, \tag{2.14b}$$

where

$$\alpha = \frac{A}{\omega^2}, \qquad \beta = \frac{2B}{\omega^2}, \gamma_1 = \frac{-3 f_3}{2\omega} \tag{2.15}$$

$$\alpha_1 = \frac{g_0 \omega^2 - A}{2\omega^2}, \qquad \delta = \frac{g_1}{2}, \; \beta_1 = 2 g_2 - \frac{3C}{2\omega^2} + \frac{2 g_1^2}{3\omega} \tag{2.16}$$

Expressing the complex-valued function ψ into real and imaginary parts, we obtain

$$\Psi(\tau) = \rho(\tau)\exp\big(i\vartheta(\tau)\big), \tag{2.17}$$

and arrive at the model equations

$$\frac{d\rho}{d\tau} = \alpha_1\rho + \beta_1\rho^3 + \delta_2\chi\rho, \tag{2.18}$$

$$\frac{d\theta}{d\tau} = \gamma_1\rho^2, \tag{2.19}$$

$$\frac{d\chi}{d\tau} = \alpha\chi + \beta\rho^2. \tag{2.20}$$

From equations (2.8), (2.10) and (2.17) we can express the field *X(t)* to the second approximation as

$X(t) = 2\varepsilon\rho(\tau)\cos\big(\omega t - \vartheta(\tau)\big) + \varepsilon^2\chi(\tau),$ (2.21)

where ρ and θ and $\chi$are given by Equations (2.18-2.19-2.20).

Moreover, the approximate solution is asymptotically exact, i.e. valid on bounded intervals of the τ-variable and on t-scale $z = O(1/\varepsilon^2)$. If one wishes to construct solutions on intervals such that $\tau = O(1/\varepsilon)$, then the higher terms must be included, because they will in general affect the solution.

Periodic solutions of the complete system described by equation (2.4) correspond to the fixed points of equations (2.18-2.20), which are obtained by the conditions $d\rho/d\tau = d\chi/d\tau = 0$.

The trivial solution is possible, but steady-state self-excitations responses exists and the equilibrium points ρ_S, χ_S are given by

$$\rho_E = \sqrt{A\frac{(g_0\omega^2 - A)}{4Ag_2\omega^2 - 3AC - 2Bg_1\omega^2 + \frac{4}{3}A\omega g_1^2}}, \tag{2.22}$$

$$\theta = \gamma_1\rho_E^2\tau + \theta_0, \tag{2.23}$$

$$\chi_E = -\left(\frac{2B}{A}\rho_E^2\right) \tag{2.24}$$

where the expression inside the root square must be non negative.

In order to establish the stability of steady state solutions, we superpose small perturbations in the amplitudes and the phases on the steady state solutions and the resulting equations are then linearized. Subsequently we consider the eigenvalues of the corresponding system of first order differential equations with constant coefficients (the Jacobian matrix). A positive real root indicates an unstable solution, whereas if the real parts of the eigenvalues are all negative then the steady state solution is stable.

The eigenvalue equation is

$$\lambda^2 - 2\lambda\rho_E(\beta + \beta_1\rho_E) - \big(2\beta\rho_E(\delta\rho_E)\big) = 0, \tag{2.25}$$

where

$$\lambda_1 = \rho_E(\beta + \beta_1\rho_E) + \sqrt{(4\alpha\delta\rho_E + \rho_E^2(\beta + \beta_1\rho_E)^2)}, \tag{2.26}$$

$$.\ \lambda_1 = \rho_E(\beta + \beta_1\rho_E) - \sqrt{(4\alpha\delta\rho_E + \rho_E^2(\beta + \beta_1\rho_E)^2)} \tag{2.27}$$

and then the condition $\lambda_1, \lambda_2 < 0$, is requested for the stability of the solution (2.22-2.24).

2. The nonlocal vibration control

We study three cases:

(i) the nonlocal vibration control with $A \neq 0$, *B=C=0*. In this case, we obtain

$$\rho_S = \frac{\sqrt{A(f_0\omega^2 - A)}}{4A\ {}_2\omega^2 + \frac{4}{3}A\omega f_1^2}, \quad \chi_S = 0. \tag{3.1}$$

We observe in Fig. 1, compared with the numerical solution, that if the control term A increases then we can reduce and eventually suppress the amplitude of the self-excitation for

$$A = f_0\omega^2. \tag{3.2}$$

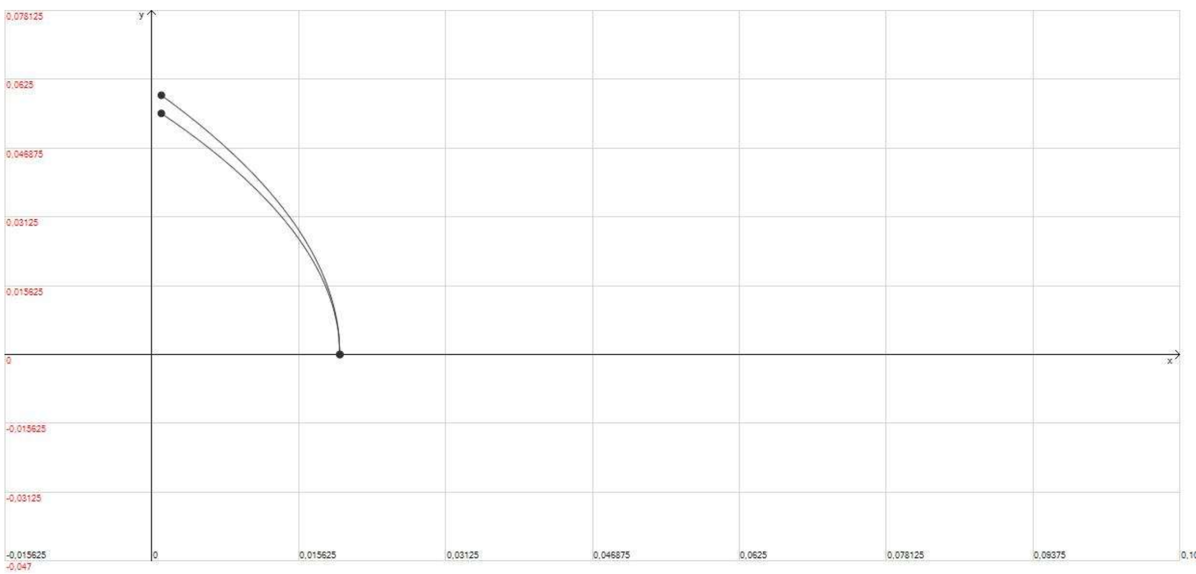

Figure 1 Stable self-excitation amplitude for the nonlocal vibration control with *g0=0.02, g1=0,9, g2=f3=1,1, ω=1, B=C=0.* The nonlocal parameter A varies from 0.001 to 0.02. The upper curve is the theoretical prevision, the lower curve comes from numerical simulation

(ii) the nonlocal vibration control with $A, B \neq 0$, *C=0*. The value of the self-excitation amplitude changes accordingly, see Fig. 2, compared with the numerical solution, but it is also present an excitation of the zero mode,

$$\rho_S = \frac{\sqrt{A(f_0\omega^2 - A)}}{4A\ {}_2\omega^2 - 2Bf_1\omega^2 + \frac{4}{3}A\omega f_1^2}, \qquad \chi_S = \frac{-2B\rho_E^2}{A}. \tag{3.3}$$

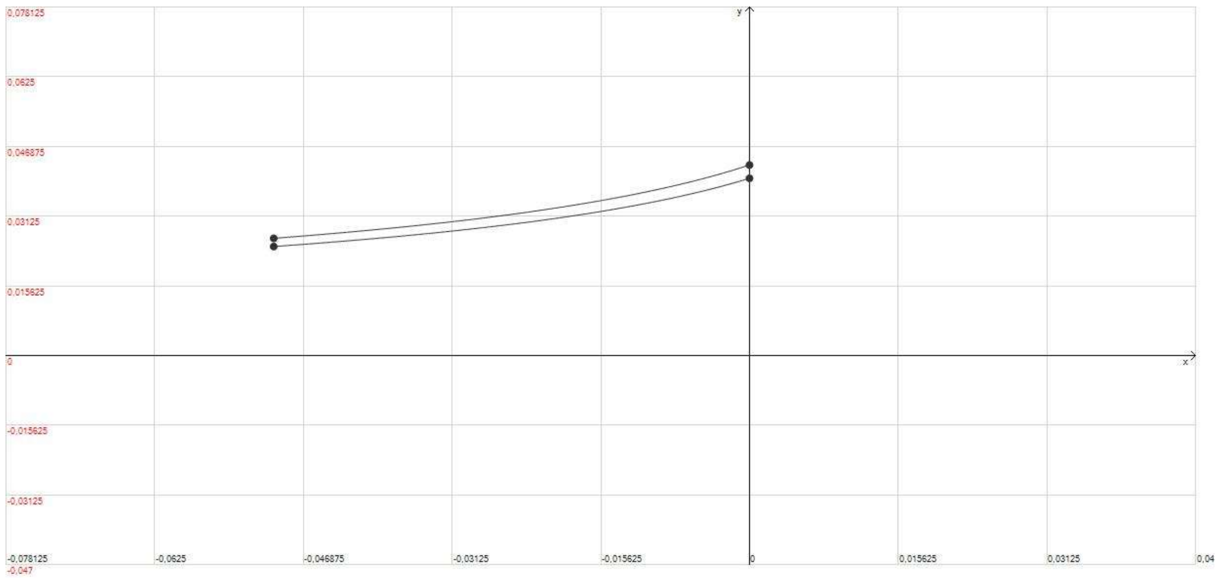

Figure 2 Stable self-excitation amplitude for the nonlocal vibration control with *g0=0.02, g1=0,9, g2=f3=1,1, ω=1, A=0,01, C=0.* The nonlocal parameter B varies from -0.05 to 0. The uppe curve is the theoretical prevision, the lower curve comes from numerical simulation

(iii) the generic delay feedback control with $A, B, C \neq 0$. In this case (see Fig. 3 for ρ_S and Fig. 4 forχ_S, both compared with numerical solution), we obtain the equations (2.22-2.24). In this case the choice (3.2) eliminates both the self-excitation ρ_S and the zero-mode amplitude χ_S, but the steady-sate solution is also dependent on the control terms B and C. The asymptotic approximate solution is

$$X(t) = 2\rho_S \cos(\omega t - \vartheta) + \chi_S, \tag{3.4}$$

where ϑ is given by (2.21)

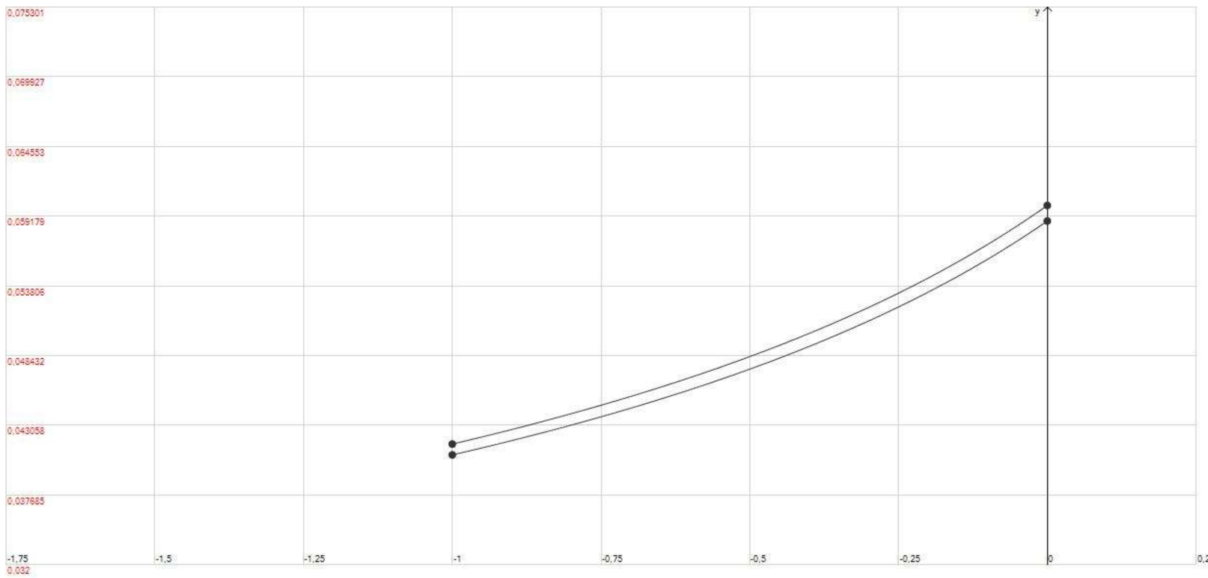

Figure 3 Stable self-excitation amplitude for the nonlocal vibration control with *g0=0.02, g1=0,9, g2=f3=1,1, ω=1, A=0,01, B=0,015.* The nonlocal parameter C varies from -2 to 0. The upper curve is the theoretical prevision, the lower curve comes from numerical simulation

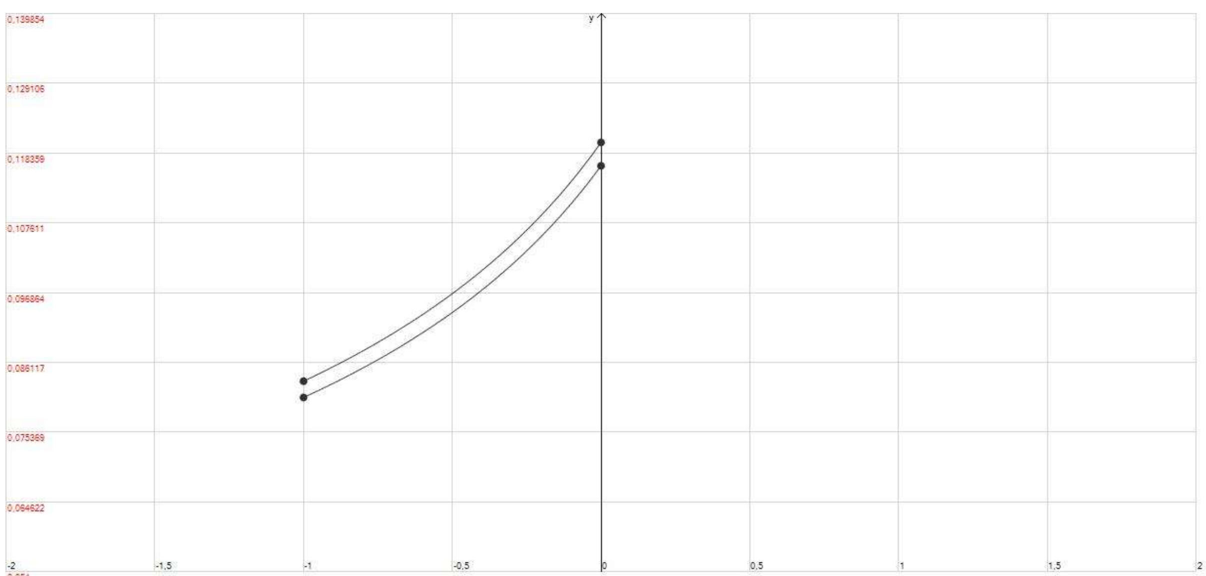

Figure 4 Stable zero-mode amplitude for the Nonlocal vibration control with *g0=0.02, g1=0,9, g2=f3=1,1, ω=1, A=0.01, B=0,015.* The nonlocal parameter C varies from -2 to 0. The upper curve is the theoretical prevision, the lower curve comes from numerical simulation

From the previous figures we conclude that appropriate choices for the nonlocal parameters can accomplish a successful control strategy for the suppression of the self-excitation of the Liènard system.

4. Conclusion

We have investigated a new method for the suppression of the self-excited vibrations based upon a nonlocal force that produces a jerk dynamics. We have considered a generic Liènard system with a nonlocal control force. Using the asymptotic perturbation method, we have obtained two slow flow equations on the amplitude and phase of the response and subsequently investigated the performance of the control strategy. We have demonstrated that the amplitude peak of the self-excitation can be suppressed and found the appropriate choices for the nonlocal parameters. This new method can be applied to the vibration control of many other nonlinear systems even with two or more degrees of freedom and we wil study this topic in future papers.

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