Fractal Oscillators

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Abstract

We consider a weakly nonlinear oscillator with a fractal forcing, given by the Weierstrass function, and use the asymptotic perturbation (AP) method to study its behavior. Being this function nowhere differentiable we can only use adequate approximations. We find that while in the linear case the resulting motion is a simple superposition between the fractal forcing and the standard oscillation, on the contrary in the nonlinear case the oscillator phase and its frequency also become fractal. We obtain the Poincarè sections in various cases and all theoretical findings are corroborated with numerical simulation.

***Keywords*:** fractal; nonlinear oscillator; perturbation method; Weierstrass function

1. Introduction

Fractals and self-similarity helped our understanding of the universe and from the Mandelbrot pioneering work they are a huge research field for physics [1-5]. In the last years a lot of research effort has been devoted to fractional oscillator processes [6-10]. In particular, two types of fractional oscillator processes, namely the Weyl or the Riemann-Liouville (RL) type, have been considered recently [6]. it is well known that the Weyl fractional oscillator process is a stationary Gaussian process. Its spectral density is given by a simple closed expression.

On the contrary, the RL-fractional oscillator process is a non-stationary process and the covariance function can be given by a complicated expression. Researchers increased their use of fractional dynamics in order to study various transport phenomena in complex and disordered media [7-10]. Our goal in this paper is to understand how the self-similarity interact with a weakly nonlinear oscillator without fractional dynamics.

We study a weakly nonlinear oscillator with quadratic and cubic nonlinearities but with a peculiar external forcing term

$$\ddot{X} + \omega^2 X = bX^2 + cX^3 + W(t) \tag{1}$$

where we have chosen a Weierstrass function $W(t)$

$$W(t) = \sum_{n=1}^{\infty} A^n \cos(B^n \Omega t). \qquad (2a)$$

It is well known that this function is continuous but nowhere differentiable if

$$0 < A < 1 \quad AB > 1 \qquad (2b)$$

The Weierstrass function with A=0.5, B= 4.0, Ω=φ=the golden mean=1.618033 989… is shown in Fig. 1.

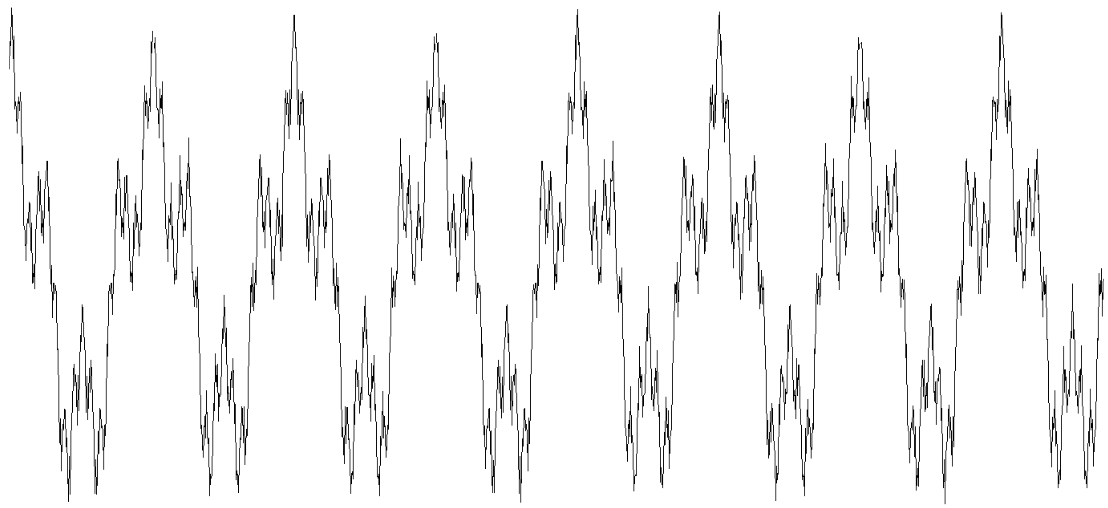

Figure 1: The Weierstrass function with A=0.5, B=4.0 N=15

In the next Section we will try to produce an approximate solution for this fractal oscillator, trying to understand the most important differences with the linear case.

2. The asymptotic perturbation method

We want to consider a weakly nonlinear oscillator with a fractal forcing, given by the Weierstrass function (1), and use the asymptotic perturbation (AP) method [11] to study its behavior. The first step for the AP method is to introduce a slow time

$$\tau = \varepsilon^q t, \qquad (3)$$

where $q > 0$ will be chosen in the following and ε is a simple book keeping device. We now build a solution of Equation (2) using this harmonics superposition

$$X(t) = (\psi_1(\tau)\exp(-i\Omega t) + \psi_0(\tau)\varepsilon + \psi_2(\tau)\varepsilon\exp(-2i\omega t) + c.c.) + O(\varepsilon^2), \qquad (4)$$

where the harmonics coefficients are depending on the slow time □.

We observe that the introduction of the slow time (3) implies

$$\frac{d}{\mathrm{dt}}\left(\psi_n \exp(-\mathrm{in}\omega t)\right) = \left(-\mathrm{in}\Omega\psi_n + \varepsilon^q \frac{d\psi_n}{dt}\right)\exp(-\mathrm{in}\omega t). \qquad (5)$$

Using Equation (4) and substituting into Equation (2) yields various equations for each harmonic *n* and for a fixed order of approximation on the perturbation parameter ε.

For *n=1*, we get

$$-2i\omega\psi_\tau\varepsilon^q = (2b)\psi_0\psi\varepsilon\varepsilon^r + (2b)\psi_2\psi_{-1}\varepsilon^2 + 3c|\psi|^2\psi\varepsilon^2 \tag{6}$$

and, after setting *q=2, r=1* for an adequate terms balance, we get for n=0, with $\Phi = \psi_0$

$$\Phi = \frac{1}{\omega^2}W(\tau) + \frac{2b}{\omega^2}|\psi|^2 \tag{7}$$

for n=2

$$A_2 = \frac{-b}{3\omega^2}\ , \ \ \psi_2 = A_2\psi^2 \tag{8}$$

Taking into account Equations (1) and (5), we can obtain a differential equation for the evolution of the complex amplitude ψ,

$$\frac{d\psi}{dt} = (i\alpha_1)\psi\Phi + (i\beta_1)|\psi|^2\psi, \tag{9}$$

with

$$\alpha_1 = \frac{b}{\omega}, \tag{10}$$

$$\beta_1 = \frac{3c}{2\omega} - \frac{b^2}{3\omega^3}. \tag{11}$$

Using the polar form,

$$\psi(\tau) = \rho(\tau)\exp\big(i\theta(\tau)\big), \tag{12}$$

into Equation (9), and separating real and imaginary parts, we easily get the following model system

$$\frac{d\rho}{dt} = 0 \tag{13}$$

$$\frac{d\theta}{dt} = \alpha_1\Phi + \beta_1\rho^2. \tag{14}$$

Considering Equations (4), (7), (8) and (12), the lowest order approximate solution of Equation (2) can be written as

$$X(t) = 2\rho cos(\omega t - \vartheta) + \Phi + 2A_2\rho^2 cos(-2\omega t + 2\theta) + O(\varepsilon^2). \tag{15}$$

and Φ is given by Equation (7).

We underline this result does not depend on the peculiar external forcing form, but the point here is that its infinite Fourier components mix up in order to produce a motion characterized by infinite frequencies. Without the nonlinear terms this result is not possible and in the linear case b=c=0 we get a pure two period quasiperiodic motion, that is a closed curve in the Poincarè section. The validity of the approximate solution should be expected to be restricted on bounded intervals of the t-variable and then on time-scale $t = O\left(\frac{1}{\varepsilon}\right)$. If one wishes to construct approximate solutions on larger intervals such that $t = O\left(\frac{1}{\varepsilon}\right)$ then the higher terms will in general affect the solution and must be included. Moreover, the approximate solution (22) should be within $O(\varepsilon)$ of the true solution on bounded intervals of the t-variable. However, we can trust excessively this approximate solution, because we neglected the fundamental resonance Fourier component of the external forcing. This component should be inserted into Equation (6) and the solution can become unstable if the external forcing is too strong.

The system (13-14) can be easily solved

$$\rho(t) = \rho_0 = const \tag{16}$$

$$\theta(t) = \beta_1 {\rho_0}^2 t + \alpha_1 \int_0^t \Phi\,(t')dt'. \tag{17}$$

We underline the integral

$$I(t) = \alpha_1 \int_0^t W\,(t')dt' \tag{18}$$

in (17) cannot be written with elementary functions and the same for the approximate solution and its derivative.

3. Numerical simulation

If we consider the oscillator (1) at the time t=k T, where T is the golden mean period 3,883022077.. seconds and k a positive integer, we can conclude that this solution is not clearly a closed curve, because there are infinite frequencies coming from the external forcing in the equation (2).

We show the Poincare sections for the fractal oscillator (1) and consider a finite number of harmonics in the Weierstrass function in equation (2a), in order to perform the numerical simulation,

$$W_S(t) = \sum_{n=1}^{N} A^n \cos(B^n \Omega t) \tag{19}$$

where N=15.

We show several remarkable cases ($Y = \dot{X}$) for the Poincarè sections for the nonlinear and fractal oscillator (1). If not specified the initial conditions are given by $X_0=Y_0=0.88$.

In Figure 2 (ω=1, b=0.0, c=0.0, A=0.5, B=4, Ω=golden mean, N=15) we represent the linear case.

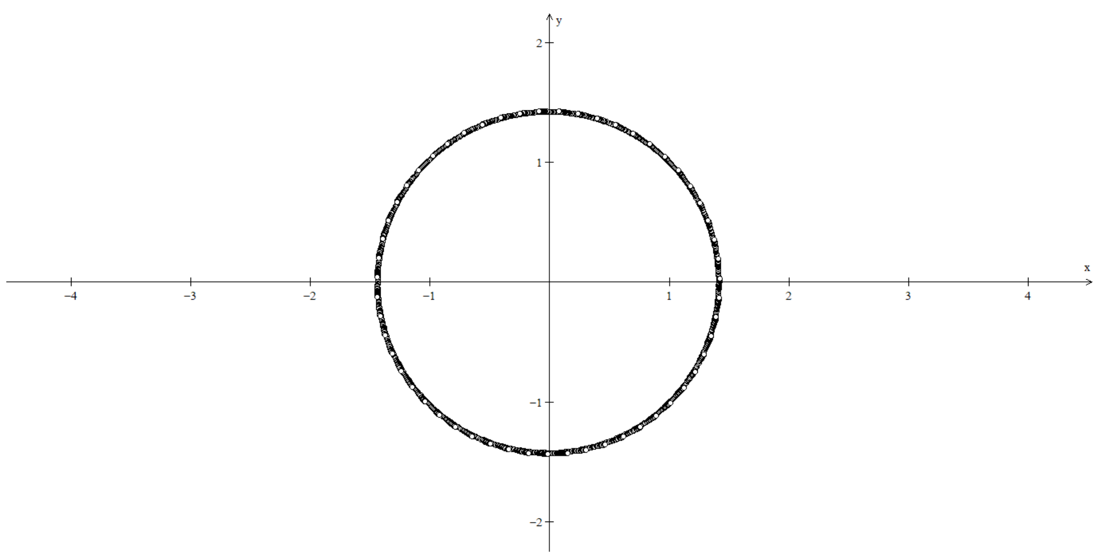

Figure 2 (ω=1, b=0.0, c=0.0, A=0.5, B=4, Ω=golden mean, N=15)

The Poincarè section is obviously a closed curve and we observe a quasiperiodic motion for the oscillator.

In Figure 3 (ω=1, b=-0.1, c=0.02, A=0.5, B=4, Ω=golden mean, N=15) we get a fractal attractor.

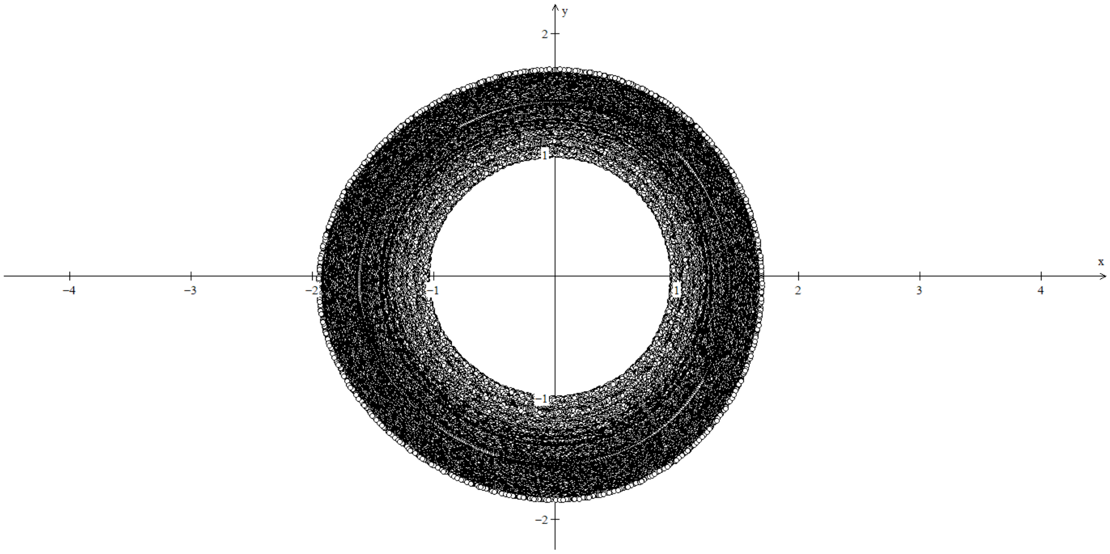

Figure 3 (ω=1, b=-0.1, c=0.02, A=0.5, B=4, Ω=golden mean, N=15)

We understand from the equation (17) that even the phase is fractal and Figure 4 is a magnified portion of the attractor shown in Fig. 3.

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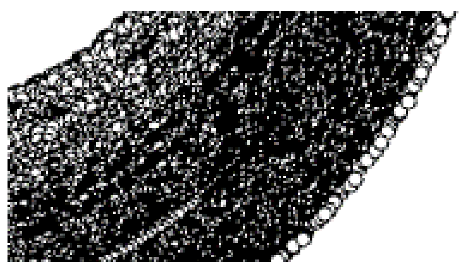

Figure 4 (ω=1, b=-0.1, c=0.02, A=0.5, B=4, Ω=golden mean, N=15)

In Figure 5 (ω=1, b=-0.1, c=0.0067, A=0.5, B=4, Ω=golden mean, N=15) we choose b and c in such a way that the coefficient β_1 given by equation (11) vanishes.

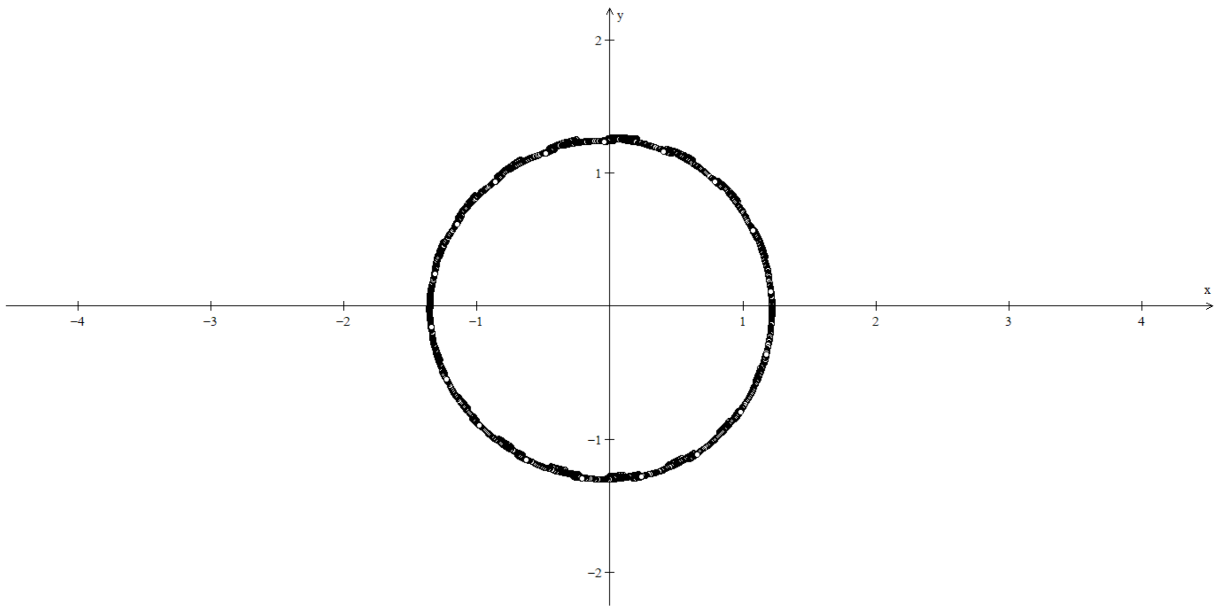

Figure 5 (ω=1, b=-0.1, c=0.0067, A=0.5, B=4, Ω=golden mean, N=15)

We conclude from equation (14) that the fractal behavior is always present.

In Figure 6 (ω=1, b=-0.1, c=0.02, A=0.5, B=4, Ω=golden mean, N=15) with initial conditions X_0=1.19, Y_0=0.

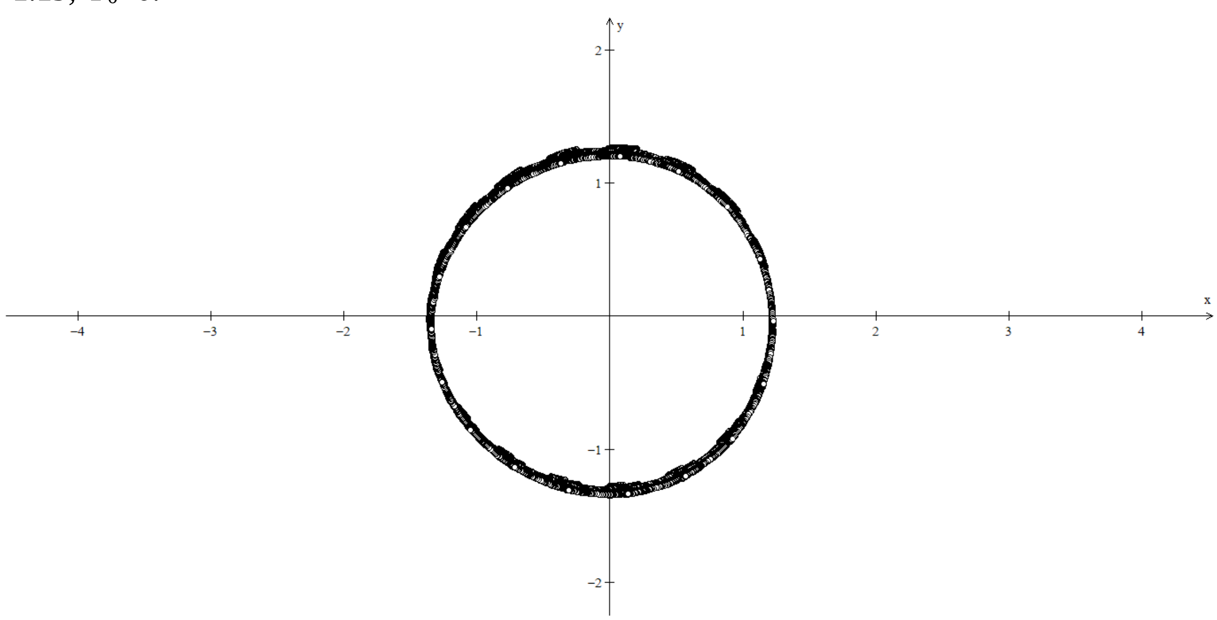

Figure 6 (ω=1, b=-0.1, c=0.02, A=0.5, B=4, Ω=golden mean, N=15)

The phase given by the equation (17) is fractal and the Poincarè section is an evenly filled phase space region but thinner than the attractor we observe in Fig. 4.

Roughly speaking we can state that numerical simulation suggests that the attractor thickness is depending on the initial conditions.

4. Conclusion

It is well known that the Weierstrass function is nowhere differentiable, with fractal properties. Following this input, we investigated a weakly nonlinear oscillator with a fractal forcing, given by the Weierstrass function, and the AP method to study its behavior. In the linear case the resulting motion is simply given by the sum of the fractal forcing with the linear oscillation, but in the nonlinear case the oscillator phase and its frequency also become fractal. We obtain the Poincarè sections in various cases and all theoretical findings are corroborated with numerical simulation. This paper could be the starting point for future research about nonlinear fractal oscillators and possible connections with real life.

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