Jerk Dynamics and Vibration Control for the parametrically excited van der Pol system

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Abstract

In parametrically excited van der Pol system, dangerous vibrations can be controlled and governed by Jerk dynamics. We choose a non-local force for the vibration control and a third or-der nonlinear differential equation (jerk dynamics) is necessary for the control method implementation. Two slow flow equations on the amplitude and phase of the response describe the oscillator motion and we are able to check the control strategy performance. The stability and response of the system is connected to the feedback gains. The dangerous excitations amplitude peak can be reduced adequately picking feedback gains. The new method is successfully checked by numerical simulation

Keywords: parametric van der Pol equation, non-local force, active control, jerk dynamics

1. Introduction

It is well known that the jerk dynamics consider the third order differential equations in the case that the dependent variable is the displacement (see Schot [1]).

Maccari [2, 3, 4] studied the non-local oscillator, i.e. an oscillator subjected to a non-local force that is equivalent to a third order differential equation using an adequate perturbation method and found the oscillator stability and its most important feature. Gottlieb [5] used the harmonic balance method and found periodic solutions and limit cycles [6-7] for some simple jerk equations.

Linz carefully investigated the relation between one–dimensional jerk dynamics and nonlinear dynamical systems in three–dimensional phase space [8, 9]. An improved balance method was used by Wu *et al.* for nonlinear jerk dynamics equations [10] .

Finally, Ma et al. [11] used the He's homotopy method and Hu [12] a first–order harmonic balance procedure with a parameter perturbation technique for nonlinear jerk dynamics.

Maccari [13] tried to figure out how the jerk dynamics can produce a novel strategy for suppressing the self-excitations for both generic nonlinear oscillators with quadratic and cubic nonlinearities and the van der Pol oscillator.

In this paper we use for the first time jerk dynamics for the vibration control and suppression of dangerous vibrations in the parametrically excited van der Pol oscillator.

A lot of effort has been devoted in the last years to vibration control. Nandi et al. [14] investigated the vibration control of a rotor using one-sided magnetic actuator and a digital proportional-derivative control. Xue and Tang [15] studied the vibration control of a nonlinear rotating beam using a piezoelectric actuator and sliding mode approach. Song and Gu [16] considered the active vibration suppression of a smart flexible beam using a sliding mode based controller.

Parametrically excited systems have been extensively studied in the last years investigated the parametric resonance control was studied by Asfar and Masoud [17] using a Lanchester-type damper and they accomplished successful vibration suppression and vibration control. Yabuno [18] considered a control law based on linear velocity feedback and linear and cubic feedback. His most important findings are that nonlinear position feedback can reduce the response amplitude in the parametric excitation-response curves and velocity feedback stabilizes the trivial solution in the frequency-response curves. A forced Duffing oscillator with time delay state feedback was investigated by Hu et al. [19] and using the multiple scales [20, 21] method they found that correct choices of feedback gains and time delay are possible for a better vibration control. Plaut and Hsieh carefully considered periodically forced nonlinear systems in the case of nonlinear structural vibrations with a time delay in damping [22] .

Maccari studied the fundamental [23] and primary [24] resonances of a van der Pol system and found that the vibration control and high amplitude response suppression can be successfully implemented using a non-local feedback control.

The van der Pol system with a non-local force is given by

$$\ddot{X} + \omega^2 X(t) - a(1 - X^2)(t)\dot{X}(t) - 2fX(t)cos(\Omega t) = F\big(X(t)\big) \tag{1}$$

where a is the coefficient of the steady source of energy ($a>0$), f the parametric excitation amplitude, $\omega \approx \frac{\Omega}{2}$ and

$$F\big(X(t)\big) = \int\big(AX(t')\big) + BX^2(t') + CX^3(t')\, dt' \tag{2}$$

is the non-local force and A, B, C appropriate control parameters (non-local feedback gains). This type of force includes the whole precedent temporal evolution of a given particle, and not only its actual position. The initial conditions are

$$X(0) = X_0 \quad \dot{X}(0) = \dot{X}_0. \tag{3}$$

The integro-differential equation (2) is equivalent to third order differential equation

$$\dddot{X}(t) + \omega^2 \dot{X}(t) - a\ddot{X}(t) + -a(2X(t)\dot{X}^2(t) + X^2(t)\ddot{X}(t)) - 2f\dot{X}(t)\cos(\Omega t) - 2f\Omega X(t)\sin(\Omega t) =$$
$$AX(t) + BX^2(t) + CX^3(t) \tag{4}$$

with the initial conditions

$$X(0) = X_0 \ . \quad \dot{X}_0 = \dot{X}(0) \tag{5}$$

$$(1+a-aX_0^2)\ddot{X}_0 = -\omega^2 \dot{X}_0 + 2f\dot{X}_0 - 2aX_0 X\dot{X}_0 \tag{6}$$

The paper is arranged as follows. In Sect. 2 we consider the van der Pol equation (1-3) and the integro-differential equation (1-2) is equivalent to third order differential equation (4).

The paper is arranged as follows. In Sect. 2 we consider the van der Pol equation (1-3) and use an adequate perturbation method [23-24], that merges together the most useful properties of harmonic balance and multiple scale methods. A slow time scale can be used to investigate the nonlinear system behaviour. A systematic means is then provided to derive increasingly accurate solutions by increasing the order of approximation in terms of a small parameter (ε). Note that, for the first-order approximate solution, results are identical to those obtainable with the other perturbation methods. Obviously, there may be other solutions, for example large-amplitude quasi-periodic motion or chaotic behavior, which the slow flow equations do not describe.

Two slow-flow equations on the amplitude and the phase are obtained. Phase-locked solutions (corresponding to periodic motion with fixed phase) and their stability are discussed. The feedback gains are chosen by analyzing the modulation equations of the amplitude and phase.

We consider how the stability and response of the system under control can be affected by appropriate choices of the feedback gains. We demonstrate that the control performance can be accomplished and the amplitude peak of the resonant response reduced.

Vibration control for the parametrically excited van der Pol system has been discussed in a previous paper [25]. The technique exposed in this paper is very different from the method discussed in [25]. We accomplish vibration control by various contour plots in such a way that we can freely choose the feedback gains that correspond to the desired amplitude response.

On the contrary, in [25] the author uses a traditional method based on parametric excitation-amplitude response curves for the uncontrolled system and the controlled system. There is no way to obtain a previously chosen amplitude response.

2. The lowest order solution

In order to apply the perturbation method we assume weak damping and (linear and quadratic) feedback gains and scale the coefficients,

$$(\mathrm{a,A,B}) \to \varepsilon^2(\mathrm{a,A,B}), \tag{7}$$

where e is a small nondimensional parameter that is artificially introduced to serve as bookkeeping device and will be set equal to unity in the final analysis.

In this section we consider the van der Pol equation with a non-local control (4) and introduce the slow variable

$$\tau = \varepsilon^2 t, \tag{8}$$

because we need to look on larger scales, in order to obtain a non-negligible contribution by nonlinear and control terms.

We consider the case of principal parametric resonance and set

$$\omega=\frac{\Omega}{2}+\varepsilon\sigma \tag{9}$$

where σ is the detunung parameter. The solution *X(t)* of equation (4) can be expressed by means of a power series in the expansion parameter ε,

$$X(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\gamma_n}\psi_n(\tau;|\varepsilon)\exp(-in\omega t), \tag{10}$$

where $g_n = |n|$ for $n^1 0$, $\gamma_0 = 2$ and $\psi_m(\xi,\varepsilon)=\psi^{-m}(\xi,\varepsilon)$.

Equation (10) can be written more explicitly

$$X(t)=\varepsilon^2\psi_0(\tau;\varepsilon)+\varepsilon(\psi_1(\tau;\varepsilon)\exp(-i\omega t)+c.c.)+h.o.t., \tag{11}$$

where *h.o.t.* = higher order terms and *c.c.* stands for complex conjugate of the preceding terms. The functions $\psi_n(\tau,\varepsilon)$ depend on the parameter ε and we suppose that the limit of the y_n for $e \to 0$ exists and is finite and moreover they can be expanded in power series of ε, i.e.

$$\psi_n(\tau;\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i\, \psi_n^{(i)}(\tau). \tag{12}$$

In the following for simplicity we use the abbreviations $\psi_n^{(0)}=\psi_n$ for $n^1 1$, $\psi_1^{(0)}=\psi$for *n=1* and $\psi_1^{(0)}=\chi$ for n=0. In the lowest order calculations, only the functions corresponding to *i=0* appear.

The solution is then a Fourier expansion in which the coefficients vary slowly in time and the lowest order terms correspond to the harmonic solution of the linear problem. Evolution equations for the amplitudes of the harmonic terms are then derived by substituting the expression of the solution into the original equations and projecting onto each Fourier mode.

For *n=1* we obtain the linear equation (order ε^2)

$$\frac{-3}{4}\Omega^2\psi_\tau + \frac{a}{4}\Omega^2(\psi - |\psi|^2\psi) - i\frac{\Omega}{2}f\psi - i\frac{\Omega^2}{2}\sigma\psi. \\ A\psi+3C|\psi|^2\psi \tag{13}$$

Considering equation (4) for *n=0* yields (order ε^3)

$$\frac{\Omega^2}{4}\chi_\tau=A\chi+2B|\psi|^2. \tag{14}$$

We can derive a nonlinear differential equation for the evolution of the complex amplitude Ψ and the real zero-mode amplitudeχ,

$$\Psi_\tau = (\alpha_1+i\alpha_2)\Psi+i\alpha_3\psi * +\beta_1|\Psi|^2\Psi, \tag{15}$$

$$\chi_\tau = (\alpha_4)\chi+\beta_2|\Psi|^2 \tag{16}$$

where

$$\alpha_1 = \frac{-4A}{3\Omega^2}+\frac{a}{3},\ \ \alpha_2 = 2\,\sigma/3\ \ ,\ \alpha_3 = -2\,f/3\,\Omega \tag{17}$$

$$\alpha_4 = \frac{4A}{\Omega^2} \qquad \beta_1 = \frac{-4C}{3\Omega^2}+\frac{a}{3}, \qquad \beta_2 = \frac{8B}{\Omega^2}. \tag{18}$$

Expressing the complex-valued function ψ into real and imaginary parts, we obtain

$$\Psi=\rho\exp(i\vartheta) \tag{19}$$

and arrive at the model equations

$$\frac{d\rho}{d\tau}=\alpha_1\rho+\beta_1\rho^3+\alpha_3\rho\sin2\vartheta, \tag{20}$$

$$\rho\frac{d\theta}{d\tau}=\alpha_2\rho-\alpha_3\rho\cos2\vartheta \tag{21}$$

$$\frac{d\chi}{d\tau}=\alpha_4\chi+\beta_2\rho^2. \tag{22}$$

From equations (20), (21) and (22) we can express the field *X(t)* to the second approximation as

$$X(t) = 2\varepsilon\rho(\tau)\cos(\omega t - \vartheta(\tau))+\varepsilon^2\chi(\tau), \tag{23}$$

where ρ and J are given by Equations (20-21).

Moreover, the approximate solution is asymptotically exact, i.e. valid on bounded intervals of the τ-variable and on *t*-scale, $t=O(1/\varepsilon^2)$. If one wishes to construct solutions on intervals such that $\tau=O(1/\varepsilon)$, then the higher terms must be included, because they will in general affect the solution.

3. Vibration control for the van der Pol equation

Periodic solutions of the complete system described by equation (4-6) correspond to the fixed points of equations (20-22), which are obtained by the conditions $d\rho/d\tau=d\vartheta/d\tau=d\chi/d\tau=0$.

The trivial solution is possible, but steady-state parametric excitation responses exist and the equilibrium points ρ_E, ϑ_E, χ_E are given by

$$\rho_E^2 = \frac{-\alpha_1}{\beta_1} \pm \sqrt{\alpha_3^2 - \alpha_2^2} = \frac{4A-a\Omega^2}{-4C+a\Omega^2} + \sqrt{4\sigma^2 - \frac{4f^2}{9\Omega^2}}, \tag{24}$$

and

$$\rho_E^2 = \frac{-\alpha_1}{\beta_1} \pm \sqrt{\alpha_3^2 - \alpha_2^2} = \frac{4A-a\Omega^2}{-4C+a\Omega^2} - \sqrt{4\sigma^2 - \frac{4f^2}{9\Omega^2}} \tag{25}$$

where the expressions $(4A - a\Omega^2)$ and $(-4C+a\Omega^2)$ must have the same sign,

$$\tan 2\vartheta_E = \frac{\alpha_1+\beta_1\rho_E^2}{\alpha_2} = \frac{-4A+a\Omega^2+\rho_E^2(a\Omega^2-4C)}{6\sigma\Omega^2} \tag{26}$$

$$\chi_E = \frac{-\beta_2\rho_S^2}{\alpha_2} = \frac{-4B}{\sigma\Omega^2}\rho_E^2. \tag{27}$$

In order to establish the stability of steady state solutions, we superpose small perturbations in the amplitudes and the phases of the steady state solutions of the equations (20-22) and the resulting equations are then linearized. We obtain from the equations (20-22) the linear equations

$$\rho_\tau=\alpha_1\rho+3\beta_1\rho_E^2\rho+\alpha_3\sin(2\theta_E)\rho+2\alpha_3\rho_E\cos(2\theta_E)\theta, \tag{27a}$$

$$\rho_E\theta_\tau=\alpha_2\rho - \alpha_3\cos(2\theta_E)\rho+2\alpha_3\rho_E\sin(2\theta_E)\theta, \tag{27b}$$

$$\chi_\tau=\alpha_4\chi+2\beta_2\rho_E\rho. \tag{27c}$$

Subsequently we consider the eigenvalues of the corresponding system of first order differential equations with constant coefficients (the Jacobian matrix). A positive real root indicates an unstable solution, whereas if the real parts of the eigenvalues are all negative then the steady state solution is stable.

The eigenvalue equation is

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0, \tag{28}$$

where

$$\lambda_1 = 2\beta_1\rho_E^2, \qquad \lambda_2 = -2\alpha_1 - 2\beta_1\rho_E^2,\ \lambda_3=\alpha_4 \tag{29}$$

and the conditions

$$\beta_1 < 0,\ \ \alpha_4 < 0\ \ \alpha_1 > -\beta_1\rho_E^2 \tag{29a}$$

are requested for the stability of the phase-locked solution (24-27). Note that the initial conditions of the van der Pol oscillator (4) are connected to the lowest order of perturbation with the initial conditions (ρ_0,ϑ_0)of the amplitude and phase through the relations

$$\rho_0 = \frac{1}{2}\sqrt{X_0^2 + \left(\frac{\dot{X}_0}{\omega^2}\right)}, \quad tan\theta_0 = \frac{\dot{X}_0}{\omega X_0}, \tag{30}$$

We study three cases for the vibration control

(i) the non-local feedback with *A0, B=C=0.* In this case, we obtain (see Figs. 1-2 for the stable(as given by (29a)) response ρ_E)

$$\rho_E^2 = \frac{-\alpha_1}{\beta_1} \pm \sqrt{\alpha_3^2 - \alpha_2^2} = \frac{4A - a\Omega^2}{a\Omega^2} \pm \sqrt{4\sigma^2 - \frac{4f^2}{9\Omega^2}}, \quad \chi_S = 0. \tag{31}$$

$$\tan 2\vartheta_E = \frac{\alpha_1 + \beta_1\rho_E^2}{\alpha_2} = \frac{-4A + a\Omega^2 + \rho_E^2 a\Omega^2}{6\sigma\Omega^2} \tag{32}$$

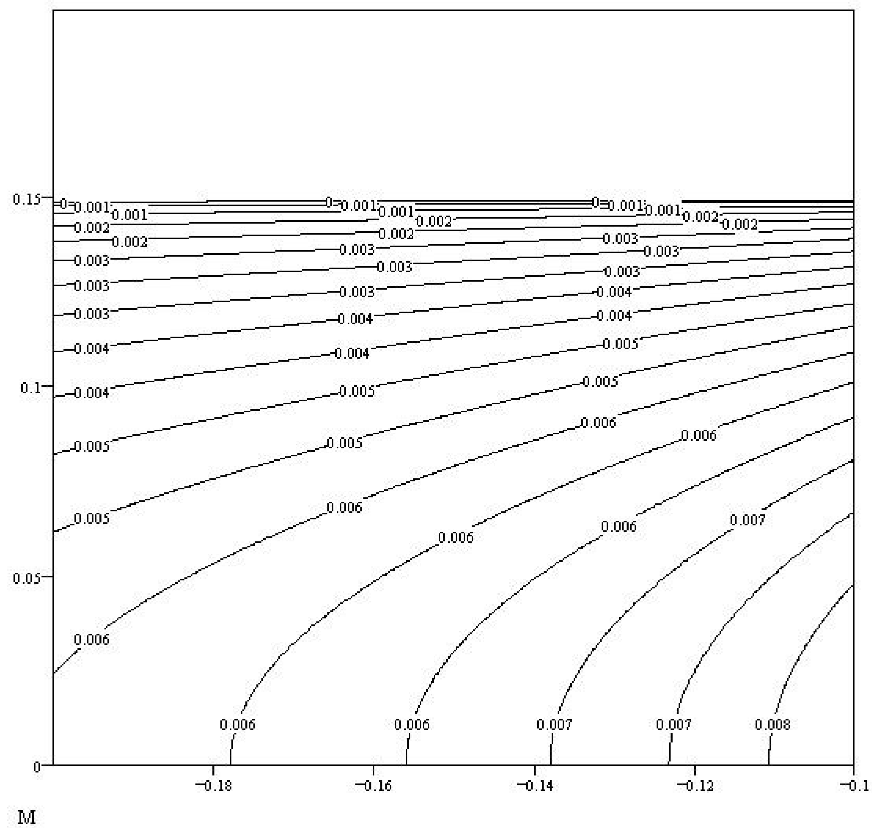

FIGURE 1: Contour plot for the stable response amplitude (24) as function of the linear feedback gain (*A*) and parametric excitation amplitude (*f*) for the van der Pol system. *A* varies from *-0.2 to 0* and *f* from *0* to *0.2*. Numbers in the plot correspond to the response amplitude (*a=0.01, B=C=0,*σ=0.05 ,$\Omega = 1$)

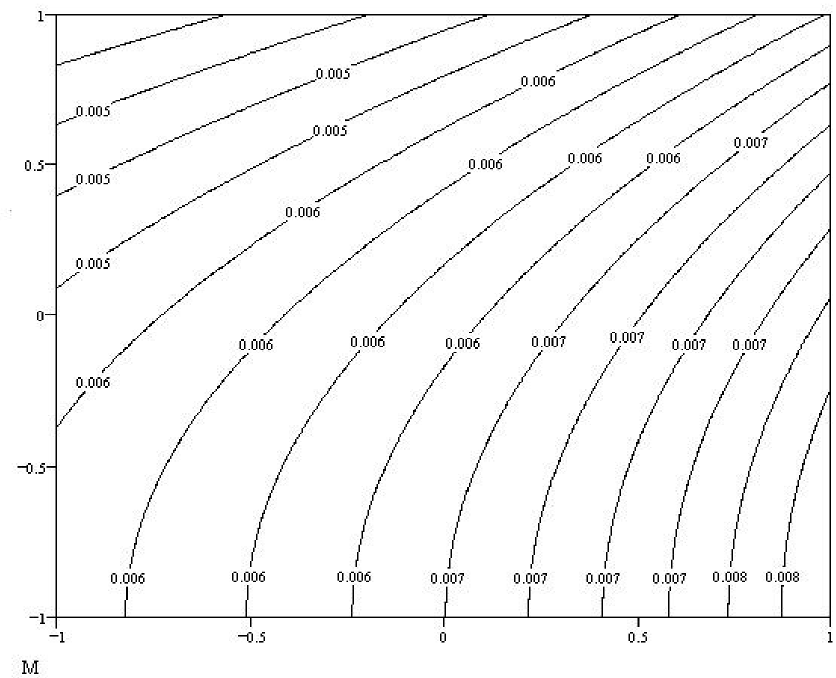

FIGURE 2: Contour plot for the stable response amplitude (25) as function of the linear feedback gain (*A*) and parametric excitation amplitude (*f*) for the van der Pol system. *A* varies from *-0.2* to *0* and *f* from *0* to *0.2***.** Numbers in the plot correspond to the response amplitude (*a=0.01, B=C=0,* $\sigma=0.05, \Omega = 1$)

We observe that we can reduce the amplitude of the parametric excitation for the following feedback gain A

$$A=\frac{a\Omega^2}{4} \pm \frac{a\Omega^2-4C}{4}\sqrt{4\sigma^2-\frac{4f^2}{9\Omega^2}}. \quad (33)$$

(ii) the non-local feedback with *A0, B0, C=0*. The value of the excitation amplitude remains given by (29-30), but it is present an excitation of the zero mode given by (27).

(iii) the generic non-local feedback with *A0, B0, C0*. In this case (see Figs. 3-4 for the stable (as given by (29a)) response ρ_E and Figs. 5-6 forχ_E), we obtain the equations (24-27).

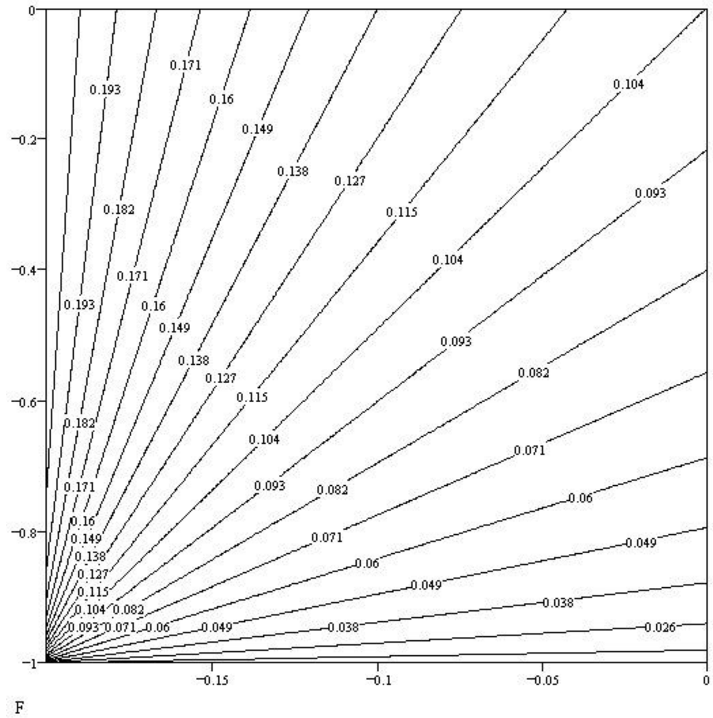

FIG 3: Contour plot for the stable response amplitude (24) as function of the linear feedback gain (A) and cubic feedback gain (C) for the van der Pol system. *A* varies from *-0.2* to *0* and *C* from *-1* to *0*. Numbers in the plot correspond to the response amplitude (*a=0.01, B=0.1,* f=0.04, $\sigma=0.05, \Omega = 1$)

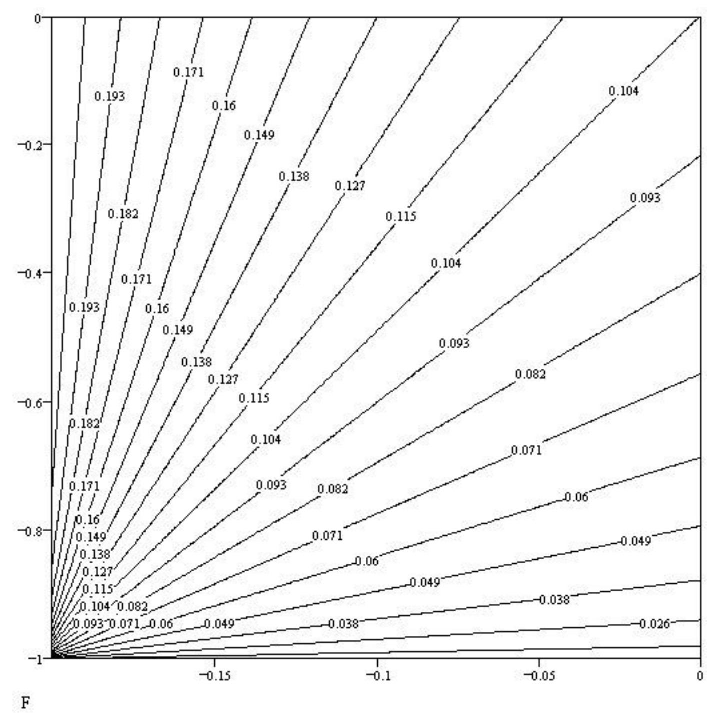

FIG 4: Contour plot for the stable response amplitude (25) as function of the linear feedback gain (A) and cubic feedback gain (C) for the van der Pol system. *A* varies from *-0.2* to *0* and *C* from *-1* to *0*. Numbers in the plot correspond to the response amplitude (*a=0.01, B=0.1,* f=0.04, $\sigma=0.05, \Omega = 1$)

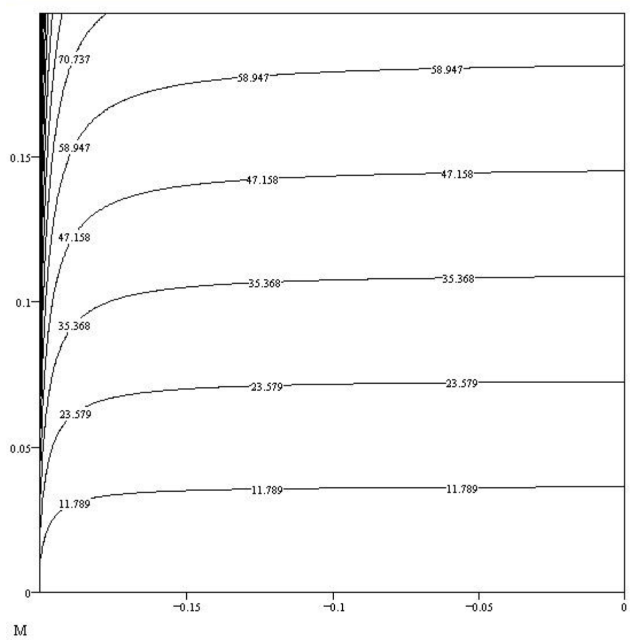

FIGURE 5: Contour plot for the zero-mode amplitude (27) with ρ_E given by (24) as function of the linear feedback gain (A) and quadratic feedback gain (B) for the van der Pol system. *A* varies from -*0.2* to *0* and *B* from *0* to *0.2*. Numbers in the plot correspond to the zero-mode amplitude (*a=0.01, f=0.04,* σ=0.05*, C=-0.01,* $\Omega = 1$)

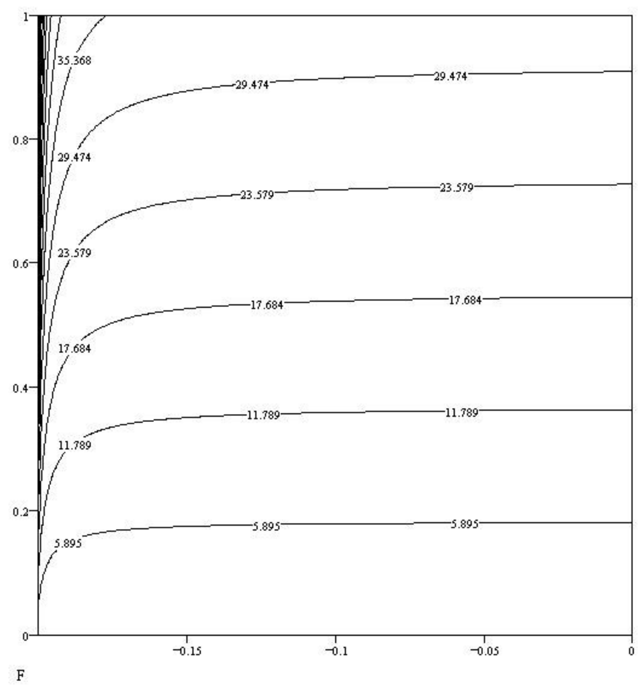

FIGURE 6: Contour plot for the zero-mode amplitude (27) with ρ_E given by (25) as function of the linear feedback gain (A) and quadratic feedback gain (B) for the van der Pol system. *A* varies from -*0.2* to *0.0* and *B* from *0* to *0.2*. Numbers in the plot correspond to the zero-mode amplitude (*a=0.01, f=0.04,* σ=0.05,*B=0.1,* $\Omega = 1$)

The choice (33) reduces both the parametric excitation amplitude ρ_E and the zero-mode amplitude χ_E, but the steady-sate solution is also reduced by the decrease of the control term B. The asymptotic approximate solution is

$$X(t) = 2\rho_E \cos(\omega t - \vartheta_E) + \chi_E, \tag{34}$$

as given by (24-27)

In order to check the validity of the control method, the van der Pol equation (4) with the initial conditions (5-6) has been numerically integrated by the Runge-Kutta-Fehlberg method. An example is given in Fig. 7. We see that there are only slight differences between the theoretical prevision (34) and the numerical result.

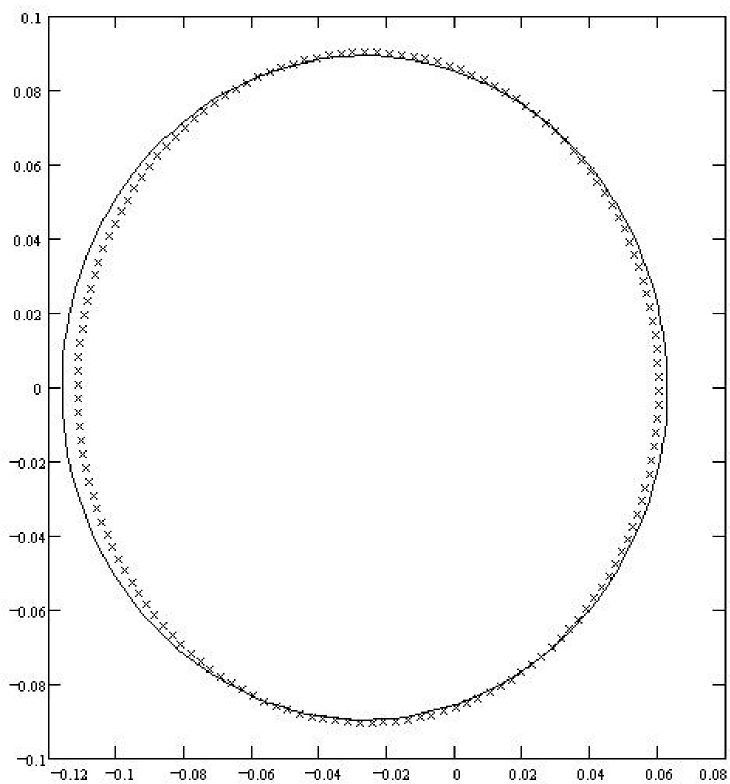

FIGURE 7: Phase-space representation of the parametrically excited van der Pol equation. Solid lines stand for the approximate solution and crosses for the numerical result. (*a=0.01,, A=-0.015, B=0.1,* C=-0.01, f=0.04, σ=0.05, $\Omega = 1$)

We conclude that appropriate choices for the feedback gains can accomplish a successful control strategy for the reduction of the parametric excitation of the van der Pol equation.

4. Conclusion

Dangerous vibrations in the parametrically excited van der Pol oscillator can be easily controlled using jerk dynamics. Taken into account that a non-local feedback force produces a jerk dynamic and using an adequate perturbation method we have derived two slow flow equations on the amplitude and phase of the response. The performance of the control strategy has been investigated and we have found that the amplitude peak of the dangerous excitation can be reduced and found the correct choices for the feedback gains. In conclusion we can state that jerk dynamics can be successfully used in order to control dangerous nonlinear vibrations, Research developments could consider laboratory experiments in order to accomplish this vibration control.

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