Isochronous and Unexpected Behavior for Complex-Valued Non-linear Oscillators with Parametric Excitation

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Abstract

Usually oscillators with periodic excitations show a periodic motion with frequency equal to the forcing one. A complex-valued nonlinear oscillator under parametric excitation is investigated by an asymptotic perturbation method based on Fourier expansion and time rescaling. Four differential equations for two nonlinearly coupled oscillators are derived. Approximate solutions are obtained and their stability is discussed. We found that the resulting motion is periodic with a frequency equal to the forcing one, if appropriate inequalities are satisfied and then for a large parameter range. The system is isochronous because periodic solutions are possible in a well-defined phase region and not only for certain discrete values. Moreover, we demonstrate that if we insert a gyroscopic term the motion can be always periodic for a well-defined parameter range but with a frequency different from the forcing frequency. Analytic approximate solutions are checked by numerical integration.

***Keywords*:** periodic motion, complex-valued system, asymptotic analysis, parametric resonance.

1. Introduction

Complex-valued nonlinear dynamical systems are very important in nonlinear dynamics and have been extensively studied [1]. Both analytical method and numerical simulation can be employed. Nonlinear rotor dynamics, high-energy particle accelerators, robots are some useful applications of complex-valued nonlinear differential equations. Colliding beams described by complex-valued nonlinear dynamical systems were investigated by Helleman [2] and Bountis and Mahmoud [3]. They demonstrated the existence and stability of periodic orbits and Mahmoud and

Aly [4] used the indicatrix method to find periodic orbits in colliding beams. Approximate analytical solutions were obtained with the generalized averaging method for periodic orbits with period equal to the damping force period (phase-locked solutions) and their stability was investigated. Complex-valued Liénard and Rayleigh equations and periodic solutions were studied by Manasevich *et al.* [5]. Another class of complex nonlinear dynamical systems was studied by Mamoud [6]. In a series of papers [7-10] Cveticanin developed a method for solving complex-valued nonlinear systems. Maccari [11] investigated period and quasi-periodic solutions of a complex-valued nonlinear system.

In the last years researchers began to consider extensively isochronous behavior but above all in autonomous systems [12-14]. A system is called isochronous when it shows in its phase space a sector where all its solutions are periodic. In this paper we will show a forced oscillator that can exhibit a periodic behavior with a frequency different from the forcing one

In this paper we study the complex-valued oscillator with parametric excitation,

$$\ddot{z}(t)+\omega^2 z(t)+(a_1+i\,a_2)\,\dot{z}(t)+b|z(t)^2|\,z(t)=2f\,z(t)\cos(2\Omega t) \qquad (1)$$

where $z(t)=x(t)+iy(t)$, the dot denotes differentiation with respect to the time, ω is the (circular) frequency, 2Ω the forcing frequency with $\Omega\approx\omega$, $a_1>0$ the dissipation coefficient, a_2 a gyroscopic parameter, b the nonlinear parameter and f the parametric excitation amplitude. The paper is arranged as follows.

In Sec. 2 we use the asymptotic perturbation (AP) method [15-16], calculate the lowest order approximate analytic solution of the equation (1) and derive a non-linear system of four coupled differential equations in the phase and amplitude of solutions.

In Sec. 3 the corresponding steady-state finite amplitude solutions are derived and we investigate the phase-locked solutions corresponding to periodic motion with frequency equal to the forcing one.

In Sec. 4 we show the most important paper finding. Periodic solutions with frequency different from the forcing frequency are possible. Analytic approximate solutions are constructed and compared with numerical integration. Final considerations are exposed in Sec. 5.

2. The asymptotic perturbation method

From the equation (1) we see that the complex-valued nonlinear system with quadratic and cubic nonlinearities corresponds to two coupled nonlinear oscillators:

$$\ddot{X}(t)+\omega^2 X(t)+a_1\dot{X}(t)-a_2\dot{Y}(t)+bX(t)(X^2(t)+Y^2(t))=2fX(t)\cos(2\Omega t) \qquad (2a)$$

$$\ddot{Y}(t)+\omega^2 Y(t)+a_1\dot{Y}(t)+a_2\dot{X}(t)+bY(t)(X^2(t)+Y^2(t))=2fY(t)\cos(2\Omega t) \qquad (2b)$$

We now introduce the slow time

$$\tau=\varepsilon t\,, \qquad (3a)$$

and the detuning parameter σ,

$$\omega = \Omega + \varepsilon\sigma, \tag{3b}$$

where ε is a bookkeeping device that will be set equal to unity in the final analysis. Using the substitution $a_1 \to \varepsilon a_1$, $a_2 \to \varepsilon a_2$, $b \to \varepsilon b$, $f \to \varepsilon f$, equation (1) yields

$$\ddot{z}(t)+\omega^2 z(t)+(a_1+i a_2)\varepsilon \dot{z}(t)+b\varepsilon|z(t)|^2 z(t)=2\varepsilon f z(t)\cos(2\Omega t) \tag{4}$$

We assume for the equation (4) a solution *z(t)* of the form

$$z(t)=\sum_{n=-\infty[odd]}^{+\infty}\varepsilon^{\gamma_n}\psi_n(\tau,\varepsilon)\exp(-in\omega t), \tag{5}$$

i.e. a power series in the expansion parameter □, with

$$\gamma_n = |n|-1. \tag{6}$$

For the system analysis not only time rescaling but also Fourier expansion are needed, because the complex oscillator reduces for a vanishing value of the parameter ε to a simple harmonic oscillator. For small values of ε, we observe a slow modulation of the coefficients of the Fourier expansion.

The assumed solution (5) can be written more explicitly

$$\begin{aligned} z(t) &= (\psi_1\exp(-i\omega t)+\varepsilon\psi_2\exp(-2i\omega t) \\ &+\psi_{-1}\exp(i\omega t)+\varepsilon\psi_{-2}\exp(2i\omega t))+\psi_0+O(\varepsilon^2), \end{aligned} \tag{7}$$

and we see that it can be considered a combination of the various harmonics with coefficients depending on □ and □. We suppose that the functions $\psi_n(\tau,\varepsilon)$ can be expanded in power series of ε, i.e.

$$\psi_n(\tau;\varepsilon)=\sum_{i=0}^{\infty}\varepsilon^i\psi_n^{(i)}(\tau). \tag{8}$$

We have assumed in equation (8) that the limit of the $\psi_n(\tau;\varepsilon)$'s for $\varepsilon \to 0$ exists and is finite. In the following for simplicity we use the abbreviations $\psi_n^{(0)}=\psi_n$ for $n \neq \pm 1$ and $\psi_1^{(0)}=\psi$, $\psi_{-1}^{(0)}=\varphi$.

Note that the introduction of the slow time (3) implies that

$$\frac{d}{dt}\left(\psi_n\exp(-in\omega t)\right)=(-in\omega\,\psi_n+\varepsilon\frac{d\psi_n}{d\tau})\exp(-in\omega t). \tag{9}$$

Using equation (5) and substituting into equation (4) yields various equations for each harmonic *n* and for a fixed order of approximation on the perturbation parameter ε.

For *n*= $\pm$ *1*, we can derive two differential equations for the evolution of the complex amplitudes ψ and φ,

$$2i\Omega\frac{d\psi}{d\tau}+\Omega(ia_1+a_2)\,\psi+2\sigma\Omega\psi+b(|\psi|^2+|\varphi|^2)\psi=f\varphi, \tag{10}$$

$$-2i\Omega\frac{d\varphi}{d\tau}+\Omega(-ia_1+a_2)\varphi+2\sigma\Omega\varphi+b(|\psi|^2+|\varphi|^2)\varphi=f\psi. \tag{11}$$

Substituting the polar forms,

$$\psi(\tau) = \rho(\tau)\exp\big(i\vartheta(\tau)\big),\ \ \varphi(\tau) = \chi(\tau)\exp\big(i\alpha(\tau)\big), \tag{12}$$

into equations (10-11), and separating real and imaginary parts, we arrive at the following nonlinear system

$$\frac{d\rho}{d\tau} = -\frac{a_1}{2}\rho + \frac{f}{2\Omega}\chi\sin(\alpha-\vartheta), \tag{13}$$

$$\frac{d\chi}{d\tau} = -\frac{a_1}{2}\chi + \frac{f}{2\Omega}\rho\sin(\alpha-\vartheta), \tag{14}$$

$$\frac{d\vartheta}{d\tau} = \sigma - \frac{a_2}{2} + \frac{b}{2\Omega}(\rho^2+\chi^2) - \frac{f\chi}{2\Omega\rho}\cos(\alpha-\vartheta), \tag{15}$$

$$\frac{d\alpha}{d\tau} = -\sigma - \frac{a_2}{2} - \frac{b}{2\Omega}(\rho^2+\chi^2) + \frac{f\rho}{2\Omega\chi}\cos(\alpha-\vartheta). \tag{16}$$

The validity of the approximate solution should be expected to be restricted on bounded intervals of the τ -variable and then on time-scale $t = O(\frac{1}{\varepsilon})$. If one wishes to construct approximate solutions on larger intervals such that $\tau = O(\frac{1}{\varepsilon})$ then the higher terms will in general affect the solution and must be included (see Section 2). Moreover, the approximate solution (7) will be within $O(\varepsilon)$ of the true solution on bounded intervals of the τ -variable, and, if the solution is periodic, for all *t*.

3. Periodic behavior with frequency equal to the forcing frequency (a_2=0)

We consider firstly the case $a_2=0$ (gyroscopic term absent). Phase-locked steady-state finite-amplitude solutions of the complex-valued nonlinear system (13-16) are given by the conditions $d\rho/d\tau = d\chi/d\tau = d\vartheta/d\tau = d\alpha/d\tau = 0$,

$$\sin\xi = \sin(\alpha_E - \vartheta_E) = \frac{\Omega a_1}{f},\ \ \rho_E = \chi_E, \tag{17a}$$

$$f = 2\Omega\sqrt{\frac{a_1^2}{4} + (\sigma + \frac{b}{\Omega}\rho_E^2)^2}. \tag{17b}$$

The first order bounded approximate solutions are

$$X = 2\rho_E\cos(\Omega t + \frac{\xi}{2})\cos(\vartheta_E + \frac{\xi}{2}), \tag{18}$$

$$Y = 2\rho_E\cos(\Omega t + \frac{\xi}{2})\sin(\vartheta_E + \frac{\xi}{2}), \tag{19}$$

or

$$z(t) = \rho_E(\exp(i(-\Omega t + \vartheta_E)) + \exp(i(\Omega t + \vartheta_E + \xi))), \tag{20}$$

Fixed-point solutions correspond to periodic solutions and we can study their stability by means of the well-known method of linearization. Small perturbations are superposed in the steady state solution and the resulting equations are then linearized.

As a consequence we calculate the eigenvalues of the Jacobian matrix that is the corresponding system of first order differential equations with constant coefficients. An unstable solution is detected by a positive real root indicates an unstable solution, otherwise if the real parts of the eigenvalues are all negative we can demonstrate that the steady state solution is stable.

We can easily find that the eigenvalues of the Jacobian matrix of the nonlinear system (35-38) are all negative for choice

$$a_1 > 0, \;\; 4b(\sigma + \frac{b\rho_E^2}{\Omega}) > 0\,. \qquad (21)$$

We underline that periodic solutions exist in a well defined parameter range then we get a new isochronous system. In Fig. 1a we show the parametric excitation-response (17b) for the case $\sigma b < 0$ ($\Omega = 1$, $a_1 = 0.1$, $a_2 = 0$, $b = 0.1$, $\sigma = -0.01$).

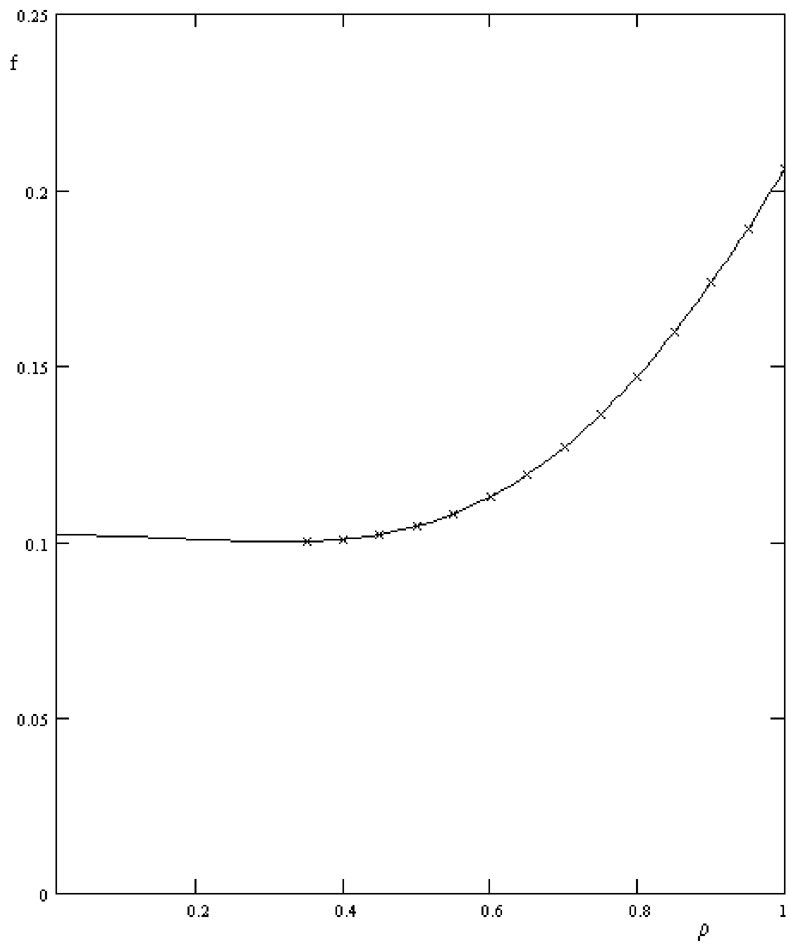

FIGURE 1a: The parametric excitation-response (17b) for the case $\sigma b < 0$. Crosses represent the stable numerical solution. ($\Omega = 1$, $a_1 = 0.1$, $a_2 = 0$, $b = 0.1$, $\sigma = -0.01$).

Stable phase-locked responses are possible only if the second condition of (21) is verified, ($\rho > 0.32$), otherwise only the trivial solution is stable. On the contrary if σ and *b* are of the same sign the condition is always verified. In Fig. 1b we show the parametric excitation-response (17b) for the case $\sigma b > 0$ ($\Omega = 1$, $a_1 = 0.1$, $a_2 = 0$, $b = 0.11$, $\sigma = 0.08$).

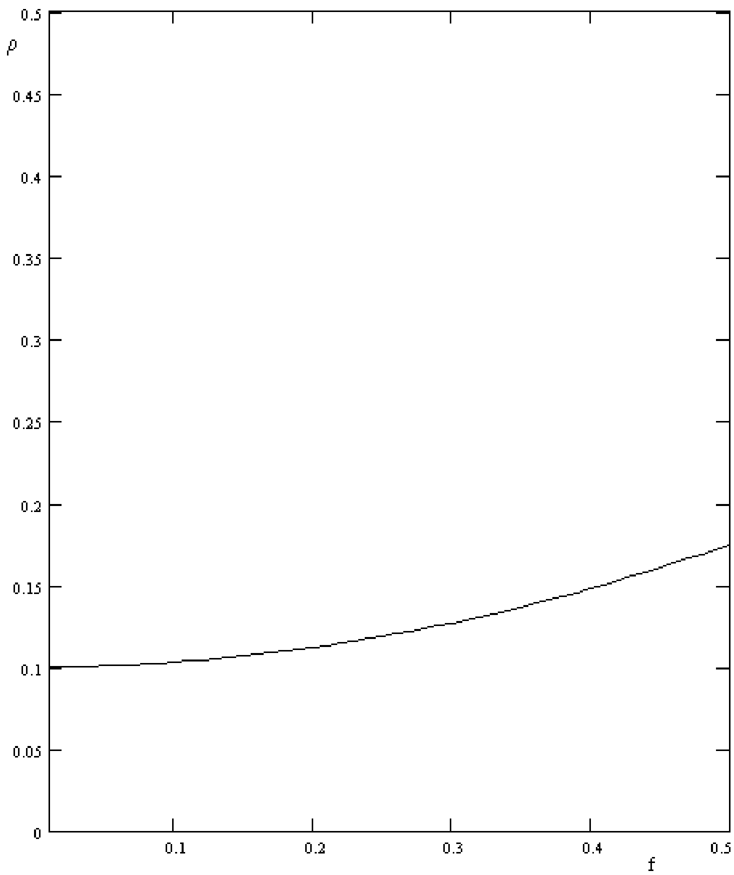

FIGURE 1b: The parametric excitation-response (17b) for the case $\sigma b > 0$. Crosses represent the stable numerical solution. ($\Omega = 1$, $a_1 = 0.1$, $a_2 = 0$, $b = 0.11$, $\sigma = 0.08$)

Crosses represent the stable numerical solution. Note that stable response are present also for small values of the excitation amplitude.

The frequency-response curve is

$$\sigma = -\frac{b\rho^2}{\Omega} \pm \frac{1}{2}\sqrt{\frac{f^2}{\Omega^2} - a_1^2} \tag{22}$$

In Fig. 2 we show the frequency-response (22) for $\Omega = 1$, $a_1 = 0.01$, $a_2 = 0$, $b = 0.1$, $f = 0.1$.

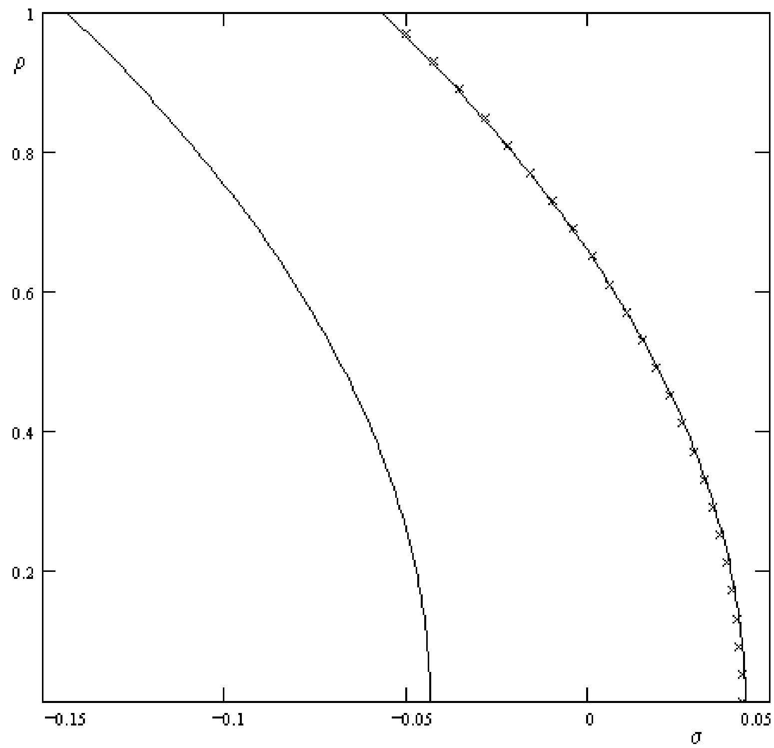

FIGURE 2: The frequency response curve (22). Note the two different branches. Crosses represent the stable numerical solution ($\Omega = 1$, $a_1 = 0.05$, $a_2 = 0$, $b = 0.1$, $f = 0.1$)

Note that the difference between the two different branches is constant and increases for increasing values of the excitation amplitude. Solutions are possible only if a threshold value is reached ($f > a_1\Omega$). The frequency increases if the response amplitude increases. Crosses represent the stable numerical solution

In Fig. 3 we show a comparison between the approximate solution (18-19) and the numerical solution of the same motion.

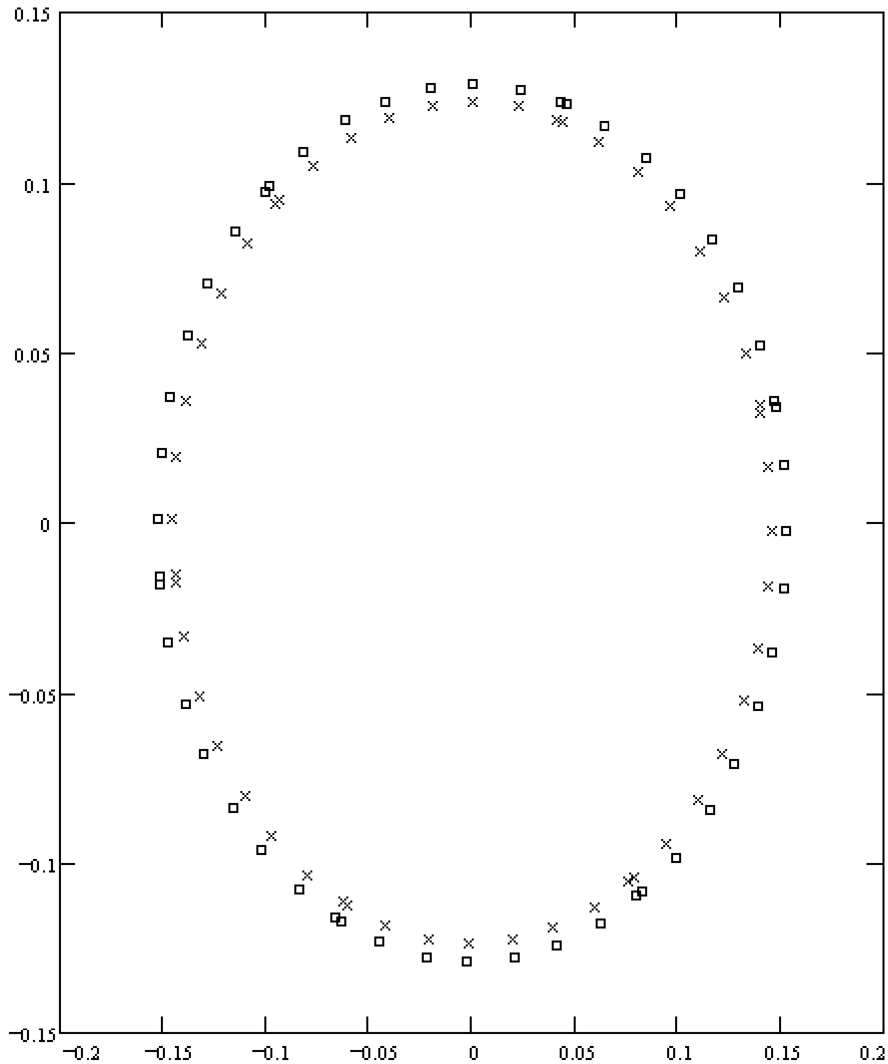

FIGURE 3: Projection on the $(X(t),\dot{X}(t))$ plane of the complex-valued system (1). Crosses are the approximate solution and boxes the numerical solution. ($\rho_E = \chi_E = 0.1$, $\Omega = 1$, $a_1 = 0.01$, $a_2 = 0$, $b = 0.15$, $f = 0.103$, $\sigma = 0.05$).

Crosses represent the approximate solution and boxes represent the numerical solution. The characteristic closed curve reveals that the motion is periodic ($\rho_E = \chi_E = 0.1$, $\Omega = 1$, $a_1 = 0.01$, $a_2 = 0$, $b = 0.15$, $f = 0.103$, $\sigma = 0.05$).

4. Periodic behavior with frequency different from the forcing one ($a_2 \neq 0$)

In this case the equations (17a-17b) are always valid but now

$$\frac{d}{d\tau}(\vartheta + \alpha) = -a_2 \tag{23}$$

Simple phase-locked solutions like in the previous section are not possible, because the solution is now

$$z(t) = \rho_E(\exp(i((-\Omega+\Omega_1)t+\vartheta_E)) + \exp(i((\Omega+\Omega_2)t)+\vartheta_E+\xi)), \tag{24}$$

where

$$\Omega_1 = -\frac{a_2}{2} + \sigma + b\rho_E^2 - \frac{f}{2\Omega}\cos\xi, \qquad \vartheta = \Omega_1 t + \vartheta_0, \tag{25}$$

$$\Omega_2 = -\frac{a_2}{2} - \sigma - b\rho_E^2 + \frac{f}{2\Omega}\cos\xi, \qquad \alpha = \Omega_2 t + \vartheta_0 + \xi. \tag{26}$$

The solution (24) is equivalent to

$$X = 2\rho_E \cos(\tilde{\Omega}t + \frac{\xi}{2})\cos(\vartheta_E + \frac{\xi}{2}), \tag{27}$$

$$Y = 2\rho_E \cos(\tilde{\Omega}t + \frac{\xi}{2})\sin(\vartheta_E + \frac{\xi}{2}). \tag{28}$$

We see that the frequency is changed and its value is

$$\tilde{\Omega} = \Omega - a_2 \quad . \tag{29}$$

For example, we consider the case with ($\rho_E = \chi_E = 0.2$, $\Omega = 1$, $\tilde{\Omega} = 1.1$, $a_1 = 0.01$, $a_2 = -0.1$, $b = 0.15$, $f = 0.112$, $\sigma = 0.05$).

We can conclude that, even if the gyroscopic term is considered, then periodic solutions are possible in a wide parameter range and we get again isochronous systems but with a slightly different frequency. In Fig. 4 we show a comparison between the approximate solution (27-28) and the numerical solution of the same motion.

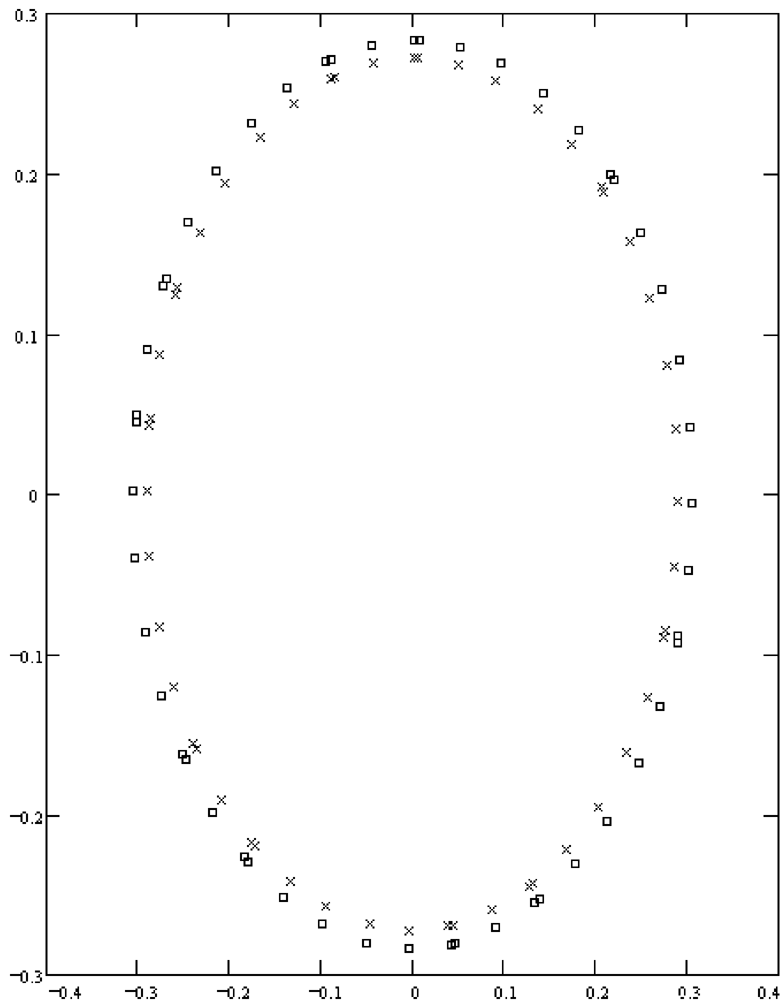

FIGURE 4: Projection on the ($X(t)$,$\dot{X}(t)$) plane of the complex-valued system (1). Crosses are the approximate solution and boxes the numerical solution. ($\rho_E = \chi_E = 0.2$, $\Omega = 1$, $a_1 = 0.01$, $a_2 = 0$, $b = 0.15$, $f = 0.112$, $\sigma = 0.05$).

Crosses represent the approximate solution and boxes represent the numerical solution. The characteristic closed curve reveals that the motion is periodic.

The agreement of the results is excellent, because the maximum difference is *0.09* and the medium difference is *0.07*, i.e. of order ε as expected.

5. Conclusion

A complex-valued nonlinear system under parametric excitation has been investigated by an asymptotic perturbation method based on Fourier expansion and time rescaling. Four coupled equations for the amplitude and the phase of solutions have been derived. We have demonstrated that the motion is periodic and shown frequency-response and parametric excitation-response curves. If we add a gyroscopic term, the motion is always periodic but with a frequency different from the forcing frequency. Analytic approximate solutions have been checked by numerical integration.

In conclusion we have corroborated the idea that isochronous systems are not rare and can possess a frequency different from the forcing one.

A direct extension of this work can be given by the introduction of other nonlinear terms or resonances (for example the fundamental 1:1 resonance).

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