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Physical model of dimensional regularization II: complex dimensions, non-scalar fields and the cosmological constant

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Abstract

In an earlier paper on the foundations of dimensional regularization, I formulated a model of a scalar quantum field whose propagator exhibits short-distance power-law screening with real positive exponent. In this paper, I heuristically generalize the model so that the propagator exhibits power-law screening at short distance with *complex* exponent. I further extend the model to Abelian gauge fields and Dirac spinors. As an unexpected byproduct, the spinor case leads to interesting extensions of the "bag" boundary conditions for the Dirac and Weyl equations. If the world really had complex dimension, it might explain in a natural way why the preponderance of observed fundamental interactions are renormalizable; and why the non-renormalizability of quantum gravity, which balances dimensional-regularization poles against a very weak coupling constant, is both acceptable and too small to be observed under ordinary circumstances. It might also motivate the famous factor of 10^{-120} between the observed cosmological constant and naïve dimensional analysis.

Keywords Dimensional regularization; complex dimension; fractal; quantum gravity; boundary conditions; cosmological constant

1. Introduction

Motivating question: In a paper on the foundations of dimensional regularization [1], I speculated that the non-renormalizability of quantum gravity is not a problem as such, but rather a fact that we manage to live with because the dimensionality of spacetime is 4-ε, with nonzero but very small ε. I went further and estimated [1] ε ~>10^{-61}. It has since struck me that this is a very interesting number, because its square is comparable to the scale of the cosmological constant problem [2] (the observed cosmological constant is smaller than what one gets from naïve dimensional analysis by a factor ~10^{-120}). I am thus led to wonder whether it might be possible to understand the cosmological constant as an O(ε^2) effect in dimensional regularization.

The answer is probably no, because too many apparently disconnected mathematical peculiarities are required. But the question is important enough that it seems worthwhile to enumerate and analyze the peculiarities anyway, and to document what can be learned by taking things as far as we can. Thus the present paper is an extended, heuristic meditation on what the detailed fractal microstructure of spacetime might have to look like so that the cosmological constant could in fact be an O(ε^2) effect. The key is to recognize that ε has to be *complex*. After that, the work amounts to trying to construct explicit if heuristic models of worlds with complex fractal dimension, and to reverse-engineer a couple of parameter constraints.

Preceding work, and broader rationale for complex dimension: Dimensional regularization [3] is generally understood as a mathematical device for turning Feynman diagram divergences into finite quantities while maintaining Lorentz and gauge invariance. It is based on two conceptual innovations. First, the four-momentum volume element d^4p is replaced by $|p|^{-\varepsilon}d^4p$, which formally resembles what the volume element would be if somehow spacetime really had dimension 4-ε. Second, divergences of order n (i.e. when the integrand decays as $|p|^{n-4}$ for large p) are defined by analytic continuation in ε starting from $\varepsilon>n$, so they depend on ε as the pole $1/(\varepsilon-n)$.Logarithmic divergences made finite in this way revert to infinity as ε approaches zero; higher-order divergences remain finite as ε approaches zero.

It could solve important problems if spacetime really did have dimension 4-ε for small $\varepsilon$and if Feynman diagrams really did take the values dictated by dimensional regularization.

- It could provide a physical origin for the observed preponderance of renormalizable interactions:If spacetime and quantum fields were literal embodiments of dimensional regularization, then quantum amplitudes would be dominated by pile-ups of logarithmic divergences (poles at ε=0), which are precisely what we call renormalizable interactions.
- It could also explain how the non-renormalizability of quantum gravity, which balances $1/\varepsilon$ poles against a very weak coupling constant, is both acceptable and too small to be observed under ordinary circumstances: As stated in[1],"Consider quantum corrections to the Einstein-Hilbert Lagrangian $(1/2\kappa^2)(-g)^{1/2}R$, where $\kappa^2=8\pi G/c^4$ is proportional to Newton's constant; g is the determinant of the metric tensor; and R is the Ricci scalar, essentially a sum of curvature components. Assume gravity is minimally coupled to matter fields but not otherwise unified with matter. The simplest induced nonrenormalizable interactions, from coupling to a massless scalar field [4] or massless photons [5], are in one loop and proportional to $(1/\varepsilon)(-g)^{1/2}Q$, where Q is quadratic in curvature components. Dimensionally,

the generic proportionality constant can only be a geometric-combinatoric number times $L_P^2/2\kappa^2$, where L_P is the Planck length. The tightest "fifth force" observational bound [6] on $R+a_2R^2$ extensions of the Einstein-Hilbert action (i.e. $Q=R^2$) is $a_2<4\text{x}10^{-9}\text{m}^2$, suggesting $\varepsilon = L_P^2/a_2 > 10^{-61}$ (ignoring geometric and combinatoric factors), easily small enough to have escaped observation."

For these reasons, it may be useful to construct explicit physical models of spacetime and quantum fields in which ultraviolet behavior is power-law screened, and superficially divergent integrals take the values dictated by dimensional regularization. In [1], I began in this direction by analyzing the ultraviolet behavior of a scalar propagator on a Euclidean "take away" fractal of dimension 3-Δ for real positive Δ, and on the Minkowski space of dimension 4-Δ obtained by tacking on a conventional time dimension.(Fractals are natural places to look for power-law behaviors because of inherent self-similarity.) I recovered ultraviolet power-law screening $|p|^{-\delta}$with $\delta=\Delta$, which provided a physical basis for regularizing logarithmic divergences with $\varepsilon=2\delta$. (Attributing power-law screening to the propagator means that power-law screening of the volume element in a loop integral is no longer fixed but depends on the number of propagators in the loop. I assume that a regularization program based on analytic continuation and poles in δ produces the same amplitudes after renormalization as one based on ε. For that reason I extend the term "dimensional" to propagator-based, not just integration-element-based, power-law screening.) But it furnished no basis for regularizing quadratic or higher-order divergences. For this reason, quite apart from the cosmological constant, I since concluded that the fractal model of [1] must be generalized to encompass a complex screening exponent. To see why, consider the schematic quadratic divergence

$$\mu^{\varepsilon}\int_m^M p^{1-\varepsilon}dp = \frac{\mu^{\varepsilon}}{2-\varepsilon}(M^{2-\varepsilon}-m^{2-\varepsilon}), \tag{1.1}$$

Where m is a physical-scale mass, M is an ultraviolet cutoff, and μ is a scale to keep overall dimensionality correct. To conform to dimensional regularization, this should reduce to $-m^{2-\varepsilon}/(2-\varepsilon)$, but that's out of the question for small real ε since the $M^{2-\varepsilon}$ term is simply too big and persistent to neglect. However, with complex ε, I conjecture that somehow oscillations in $M^{2-\varepsilon}$ average away for large M. The present paper attempts to generalize the work in[1] to scalar-field propagator power-law screening with complex exponent, and to extend this generalization from scalar fields to Dirac spinors and Abelian gauge vector fields.

[Superficially, complex-exponent power-law screening makes no sense for an expectation value of a product of real fields. But I will also modify the Boson free-field Lagrangians to have small imaginary parts, so there will be no mathematical inconsistency. Non-Hermitian time evolution may be philosophically troubling because it seems to require probability non-conservation in the form of quantum-state square-norm non-conservation; but elsewhere [7] I present arguments against axiomatically identifying square norm with probability.]

Cosmological constant: The analog of Eq. (1.1) for the cosmological constant is

$$\mu^{\varepsilon}\int_{m}^{M} p^{3-\varepsilon}dp = \frac{\mu^{\varepsilon}}{4-\varepsilon}(M^{4-\varepsilon} - m^{4-\varepsilon}). \tag{1.2}$$

If we assume the M term on the right hand side somehow averages away as before, then for small ε this reduces to

$$\frac{-\mu^{\varepsilon}m^{4-\varepsilon}}{4-\varepsilon} \sim \frac{m^4}{4}\left\{1+\varepsilon\left[\frac{1}{4}+\ln\left(\frac{\mu}{m}\right)\right]+\varepsilon^2\left[-\frac{1}{16}+\frac{1}{4}\ln\left(\frac{\mu}{m}\right)+\frac{1}{2}\left(\ln\left(\frac{\mu}{m}\right)\right)^2\right]\right\}. \tag{1.3}$$

Since Fermions and Bosons contribute to the cosmological constant with distinct signs s, we see that Eq. (1.3) leads to an $O(\varepsilon^2)$ cosmological constant *if* the following constraints hold:

$$\sum_i s_i m_i^4 = 0, \qquad \sum_i s_i m_i^4 \ln(m_i) = 0, \tag{1.4}$$

in an obvious notation. If the first equation holds, the value of the sum in the second equation is independent of how the m_i are normalized. I do not speculate about how constraints (1.4) might a rise.

The rest of this paper: In the next section, I review concepts, techniques and results from [1]. In Section 3, I generalize the thought process to formulate explicit, if heuristic, scalar-field examples of complex-exponent power-law screening. In Sections 4 and 5, I extend the thought process further to gauge and Dirac spinor fields, respectively. An obvious next step, not treated here, would be extending further to the propagator for linearized gravity.

2. Real-exponent power-law screening

In Ref. [1] I developed a construction in which scalar propagators in fractal spacetimes exhibit power-law screening of the form $|p|^{-\delta}$ in momentum space for large momentum p and nonzero δ, or, as appropriate, $|x|^{+\delta}$ in position space for small position x.

I constructed a fractal spacetime by self-similarly removing a sequence of spheres from Euclidean 3-space and then tacking on a conventional time dimension. This construction is manifestly not Lorentz invariant, but I argued that the Lorentz non-invariance would make no difference in quantum amplitudes for small dimension deficit (and that in any event there is no such thing as a Lorentz-invariant fractal construction for spacetime dimension greater than 2). The construction was based both on geometric principles and on consideration of the boundary conditions for the scalar field at the sphere surfaces.

The geometrical part of the construction defined a random "take-away" fractal, a set formed by the following recursive procedure: Start with a linear space of integer dimension D, and a reference block (most conveniently a sphere) of volume V. Distribute points randomly throughout space with arbitrary density ρ, and, centered at each such point, remove a copy of the reference block. Call this the zeroth iteration. Now choose an arbitrary scale factor $\xi>1$ and define the k'th iteration inductively as follows:

- Distribute points randomly with density $\rho\xi^{Dk}$ throughout whatever part of the Euclidean space has not been removed by preceding iterations.
- Centered at each such point, remove from the k-1'st iteration a copy of the reference block linearly scaled by factor ξ^{-k}.

In the limit of infinite k, what's left has fractal dimension $D+\ln(1-\rho V)/\ln\xi$[8]. The factor $(1-\rho V)$ is the volumetric proportion of iteration k-1 removed by iteration k, and the ratio of logarithms is minus a dimension deficit for physical $\rho V<1$.

At the surfaces of removed blocks, I assumed the scalar field ϕ obeys either Neumann boundary conditions

$$\frac{\partial\phi}{\partial n}=0, \tag{2.1}$$

or zero-enclosed-charge Dirichlet, i.e. ϕ = constant supplemented by

$$\oiint\frac{\partial\phi}{\partial n}=0. \tag{2.2}$$

Under either rule, any solution of the wave equation extremizes the usual free-particle Lagrangian restricted to the limiting take-away fractal, and (see extended parenthesis at end of the next section) has the virtue of preventing an external field (such as that from the charge at the source of a propagator) from inducing bare monopole charges inside the removed blocks.

To analyze propagator screening at small distance in a take-away fractal embedded in Euclidean 3-space, I standardized the reference block to a sphere. I then noted that any sphere removed at iteration k acts on the scalar field as an induced dipole moment with polarizability $\gamma_k\equiv(3V/4\pi\xi^{3k})g$ = $-3V/8\pi\xi^{3k}$ for Neumann and $3V/4\pi\xi^{3k}$ for Dirichlet. I then adapted dielectric theory to argue that the spheres at iteration k collectively amplify or screen a distant charge by a factor

$$\Phi_k = \left[1 + \frac{4\pi\rho\xi^{3k}\gamma_k}{1 - \frac{4\pi}{3}\rho\xi^{3k}\gamma_k}\right]^{-1} = \left[\frac{1 - \rho V g}{1 + 2\rho V g}\right] \tag{2.3}$$

Each iteration of the fractal process multiplies the Green's function (potential) of a point charge by this factor in the space between spheres, *but only for iterations whose spheres are smaller than the distance to the point charge*, since larger spheres don't fit. At the same time, each iteration also multiplies the point-charge potential by a factor of (1-ρV) for integration volume regardless of sphere size. In other words, the Green's function for point charge q becomes

$$-\frac{q}{r}\prod_{k=0}^{\infty}(1 - \varrho V)\Phi_k \prod_{l=0}^{l_{max}} \Phi_l{}^{-1} \tag{2.4}$$

Where l_{max} is the highest iteration whose spheres are larger than or equal to r. The first (infinite) product is independent of r, and so can be absorbed into an overall scale factor. (Alternatively, as in [1], we could fine-tune the proportion of Neumann and Dirichlet so that on average the multiplicand is unity for all k.) Since l_{max} satisfies r ~ radius of iteration-l_{max} sphere (proportional to $V^{1/3}/\xi^{l_{max}}$), expression (2.4) amounts to power-law screening

$$\left(\frac{r}{V^{1/3}}\right)^{-\ln(1-3\rho V/2)/ln\xi} \sim \left(\frac{r}{V^{1/3}}\right)^{3\rho V/2ln\xi} \tag{2.5}$$

for pure Neumann (g = -1/2) and small ρV.The exponent in expression (2.5) is three-halves the actual dimension deficit (the exponent would equal the actual dimension deficit for the fine-tuned Neumann-Dirichlet blend).The reader may consult [1] directly for extension to Minkowski space.

3. Complex-exponent power-law screening

If we persist in assuming the field can't penetrate the blocks we've removed, then we're stuck with Neumann or Dirichlet boundary conditions (see extended parenthesis at the end of this section), which means we're stuck with real polarizabilities in either case, and obviously also real polarizabilities if the distribution of blocks is a blend of the two. That means, following Equation (2.5), we're also stuck with a real screening power-law exponent.

One way to add more flexibility to the polarizability without inducing bare monopole charges in the blocks (see extended parenthesis at the end of this section) begins by noting that the discussion in Section 2 implicitly deals not with the usual Lagrangian for Poisson's equation, but instead (for spatial variation only) with the modification

$$\mathcal{L} = \int (B_0 \nabla\phi \cdot \nabla\phi), \tag{3.1}$$

Where B_0 is a binary function of position that takes the value unity on the take-away fractal and zero elsewhere (a useful product representation of B_0 is introduced and analyzed in [9]). To open new possibilities, let us generalize this Lagrangian to

$$\mathcal{L} = \int \left(B_\zeta \nabla\phi \cdot \nabla\phi\right), \tag{3.2}$$

where ζ is a new parameter and $B_\zeta \equiv [1+(B_0-1)(1-\zeta)]$, i.e. a function of position equal to unity inside the take-away fractal (outside the blocks) and ζ outside (inside the blocks).Thus we allow for some field leakage from the original limit fractal into its surroundings (i.e. into the blocks). Boundary conditions at the surfaces of blocks then become continuity conditions

$$\phi_{out} = \phi_{in}, \left.\frac{\partial\phi}{\partial n}\right|_{out} = \zeta \left.\frac{\partial\phi}{\partial n}\right|_{in}, \tag{3.3}$$

which generalizes Neumann, or

$$\phi_{out} = \zeta\phi_{in}, \left.\frac{\partial\phi}{\partial n}\right|_{out} = \left.\frac{\partial\phi}{\partial n}\right|_{in}, \tag{3.4}$$

which generalizes Dirichlet. For nonzero ζ, Equations (3.3) and (3.4) imply no charge can be externally induced in a block. For spherical blocks, Equations(3.3) and (3.4) lead to polarizability factors

$$g = -\left(\frac{1-\zeta}{2+\zeta}\right) \text{ and } -\left(\frac{1-\zeta^{-1}}{2+\zeta^{-1}}\right), \tag{3.5}$$

respectively. [The apparent symmetry $\zeta \leftrightarrow \zeta^{-1}$is not deep. It merely reflects triviality of the underlying ansatz: plane wave plus dipole field outside the sphere and solely plane wave inside.] For ζ small and imaginary, $\zeta \equiv i\chi$, the Neumann option turns the Equation (2.5)'s small-distance power-law screening into

$$\left(\frac{r}{V^{1/3}}\right)^{(1-i\chi)3\rho V/2ln\xi}, \tag{3.6}$$

When source and test points for the propagator are both outside all removed spheres, i.e. both inside the take-away fractal. This is the screening that one associates with dimensional regularization for complex dimension, but it doesn't apply everywhere because there's no screening if both source and test point are inside the same sphere. I can arrange for Eq. (3.6) to be the only screening regime that matters physically by constraining interaction terms such as ϕ^4 to be masked in the Lagrangian by B_0 rather than B_ζ. This is certainly contrived; nonetheless it is a first example of an explicit model of a quantum field of any kind that embodies dimensional regularization with complex dimension. Considerations in [1] can readily be adapted to argue that the propagator extended to Minkowski space also exhibits the same power-law screening.

[One way to introduce an imaginary parameter without allowing field leakage into take-away blocks is to add a block-surface term

$$\frac{\beta}{2}\oiint \phi^2 \tag{3.7}$$

to the free-particle Lagrangian. In this case, Neumann would generalize to

$$\frac{\partial\phi}{\partial n} + \beta\phi = 0, \tag{3.8}$$

(Robin boundary condition [10]) and zero-enclosed-charge Dirichlet would generalize to ϕ = constant supplemented by

$$\oiint \left(\frac{\partial\phi}{\partial n} + \beta\phi\right) = 0, \tag{3.9}$$

where normal derivative points *into* the removed block. Either of these possibilities is problematic because an external field, such as that from the charge at the source of a propagator, would induce the field of a bare monopole charge emanating from a removed block, and I don't see how that can generate an overall response as benign as power-law screening.]

4. Abelian gauge field

In this case the analogue of Lagrangian (8) is

$$\mathcal{L} = \int\left(B_\zeta[|\mathbf{E}|^2 - |\mathbf{B}|^2]\right), \tag{4.1}$$

where as usual **E** and **B** are electric and magnetic field. For time-independent fields in Euclidean 3-space, extremizing (4.1) entails, as usual, setting $\mathbf{E}=\nabla\phi$ and $\mathbf{B}=\nabla\times\mathbf{A}$. If we don't care (but of course we do) about generalizing to time-dependent electromagnetism, this system is much easier to analyze with the ansatz $\mathbf{B}=\nabla\phi_m$ instead. Then each of ϕ and ϕ_m separately satisfies Equations(3.3) or (3.4), and complex-power-law shielding applies separately to the propagators

of ϕ and ϕ_m. (Since there are no magnetic monopoles, it is a dipole singularity in the ϕ_m propagator that's screened by the mechanism of Section 3.)
In Coulomb gauge, this is easy to translate into vector-potential power-law screening. Let the potential of a magnetic monopole source (including power-law screening) be θ with $\nabla^2\theta=0$ in the take-away fractal and away from the source. Then the potential of a dipole source takes the form $\mathbf{d}\cdot\nabla\theta$ and the magnetic field is

$$\mathbf{B} = \boldsymbol{\nabla}(\mathbf{d}\cdot\boldsymbol{\nabla}\theta) = (\mathbf{d}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\theta = \boldsymbol{\nabla}\times(-\mathbf{d}\times\boldsymbol{\nabla}\theta) \tag{4.2}$$

with $\nabla\cdot(-\mathbf{d}\times\nabla\theta) = 0$. So if the magnetic dipole potential is subject to power-law screening, then so is the corresponding Coulomb-gauge vector potential restricted to the take-away fractal.

5. Dirac spinor field

In order to construct a spinor propagator with power-law screening, one may be tempted to start by observing that a solution ψ of the massless Dirac equation can be expressed as follows in terms of an underlying four-component complex scalar potential Ψ that satisfies Poisson's equation away from the source singularity

$$\psi = \boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\Psi, \tag{5.1}$$

where the components of the spatial vector $\boldsymbol{\gamma}$ are the Dirac matrices. Then one could apply boundary conditions (2.1) or (2.2) or (3.3) or (3.4) to Ψ and proceed to power-law screening as in Sections 2 or 3. But this won't work because boundary conditions that arise from variation of the Dirac Lagrangian refer only to simple values of the spinor field ψ and there is just no productive way to distill values or derivatives of Ψ from them. The boundary constraint from varying the Dirac Lagrangian with a hard surface is

$$\oiint \psi^\dagger\gamma^{\mathbf{0}}(\mathbf{n}\cdot\boldsymbol{\gamma})\delta = 0, \tag{5.2}$$

and the boundary constraint from varying the Dirac Lagrangian with a surface of discontinuity is

$$\oiint \left[\psi_{out}{}^\dagger\gamma^{\mathbf{0}}(\mathbf{n}\cdot\boldsymbol{\gamma})\delta_{out} - \psi_{in}{}^\dagger\gamma^{\mathbf{0}}(\mathbf{n}\cdot\boldsymbol{\gamma})\delta_{in}\right] = 0. \tag{5.3}$$

A famous way to ensure constraint (5.2)is the "bag" boundary condition [11]; it requires that ψ and δ at the boundary are eigenvectors of i($\mathbf{n}\cdot\boldsymbol{\gamma}$) with the same eigenvalue, i.e. ψ at the boundary can be expressed in terms of some other spinor ψ' as

$$\psi = (1 \pm i\mathbf{n}\cdot\boldsymbol{\gamma})\psi' \tag{5.4}$$

and similarly for δ. The analogue of this that applies to Equation (5.3) is

$$\psi_{out} = (a + b\mathbf{n}\cdot\boldsymbol{\gamma})\psi_{in}, \tag{5.5}$$

Where a and b are both real, can both depend on position on the boundary, and $a^2+b^2=1$. Unfortunately, neither of these boundary conditions can lead to the desired power-law screening because if the surface is a sphere, then $\mathbf{n}\cdot\boldsymbol{\gamma}$ looks like the spinor field from a point source (i.e. Equation (5.1) with $\Psi \propto 1/r$) and we have already conceded the difficulty of obtaining power-law screening from induced bare point charges.

Let us instead seek different, more favorable boundary conditions that satisfy constraints (5.2) and (5.3). The bag condition is a special case of a more general statement: ψ is an eigenvector of a 4x4 Hermitian matrix M that anticommutes with $\gamma^0(\mathbf{n}\cdot\boldsymbol{\gamma})$. There are in fact 8 such linearly independent matrices: one can write in general

$$M = a_1\gamma^0 + a_2 \mathrm{i}\mathbf{n}\cdot\boldsymbol{\gamma} + a_3\gamma^0\mathbf{n}_\perp\cdot\boldsymbol{\gamma} + a_4\gamma^0\mathbf{n}_{\perp'}\cdot\boldsymbol{\gamma} + a_5 i\gamma^5\gamma^0 + a_6\gamma^5\mathbf{n}\cdot\boldsymbol{\gamma} + a_7\gamma^5\gamma^0\mathbf{n}_\perp\cdot\boldsymbol{\gamma} + a_8\gamma^5\gamma^0\mathbf{n}_{\perp'}\cdot\boldsymbol{\gamma}, \tag{5.6}$$

Where the a_i are real and can depend on position on the surface, and the unit vectors

$$\mathbf{n}_\perp \equiv \left(\mathbf{k} - \mathbf{n}(\mathbf{k}\cdot\mathbf{n})\right)/\sqrt{|\mathbf{k}|^2 - (\mathbf{k}\cdot\mathbf{n})^2}, \mathbf{n}_{\perp'} \equiv \frac{(\mathbf{k}\times\mathbf{n})}{\sqrt{|\mathbf{k}|^2-(\mathbf{k}\cdot\mathbf{n})^2}} \tag{5.7}$$

for some arbitrary vector $\mathbf{k}$. [As this paper was going into publication, I became aware of Reference [12], which performs a similar analysis for the case δ=0]. This is more promising for power-law screening because the first definition in Equation (5.7) has ingredients that are familiar from a generic dipole field. The anticommutation condition defining M implies M's eigenvalues come in ± pairs, so Equation (5.4) generalizes to

$$\psi = (\lambda \pm M)\psi' \tag{5.8}$$

when there are only two distinct eigenvalues $\pm\lambda$. When there are four distinct eigen values $\pm\lambda_1$ and $\pm\lambda_2$, then Equation (5.8) must be replaced by

$$\psi = \left(\lambda_1{}^2 - M^2\right)(\lambda_2 \pm M)\psi' \tag{5.9}$$

Or the same thing with 1↔2.

Equation (5.5) is also a special case of a more general statement that turns out, as explained later, to be more convenient for our purposes:

$$\psi_{out} = (a + bN)\psi_{in}, \tag{5.10}$$

where the 4x4 matrix N is antihermitian, commutes with $\gamma^0(\mathbf{n}\cdot\gamma)$, and $N^2 = -1$; and a and b are both real, can depend on position on the surface, and $a^2+b^2=1$. The general solution for N has zero matrix elements connecting the +1 and -1 eigenspaces of $\gamma^0(\mathbf{n}\cdot\gamma)$, i.e. within each such eigenspace it's a 2x2 antihermitian submatrix with square **-1**. Any such submatrix can only be $\pm i\mathbf{1}$ or $i\mathbf{v}\cdot\boldsymbol{\sigma}$ with $\mathbf{v}$ real and $|\mathbf{v}|=1$. In terms of the usual Dirac matrices this amounts to

$$N = i\gamma^0\mathbf{n}\cdot\boldsymbol{\gamma} \equiv iP, \text{ or } i\mathbf{1}, \text{ or}$$

$$i\left(\frac{\mathbf{1}\mp P}{2}\right)\mathbf{1} + i\left(\frac{\mathbf{1}\pm P}{2}\right)(\mathrm{v}_1\gamma^5 + \mathrm{v}_2 i\mathbf{n}_\perp\cdot\boldsymbol{\gamma} + \mathrm{v}_3 i\mathbf{n}_{\perp'}\cdot\boldsymbol{\gamma})\text{or}$$

$$i\left(\frac{\mathbf{1}-P}{2}\right)(\mathrm{v}_1\gamma^5 + \mathrm{v}_2 i\mathbf{n}_\perp\cdot\boldsymbol{\gamma} + \mathrm{v}_3 i\mathbf{n}_{\perp'}\cdot\boldsymbol{\gamma}) + i\left(\frac{\mathbf{1}+P}{2}\right)(\mathrm{v}'_1\gamma^5 + \mathrm{v}'_2 i\mathbf{n}_\perp\cdot\boldsymbol{\gamma} + \mathrm{v}'_3 i\mathbf{n}_{\perp'}\cdot\boldsymbol{\gamma}), \tag{5.11}$$

where the v's and v-primes are all real, can all depend on position on the surface, and $|\mathbf{v}| = |\mathbf{v}'| = 1$.

[For the Weyl equation for a two-component spinor $((\nabla\cdot\boldsymbol{\sigma}-\partial_t)\psi=0$ where the components of $\boldsymbol{\sigma}$ are the Pauli matrices) the analog of M is a Hermitian 2x2 matrix that anticommutes with $\mathbf{n}\cdot\boldsymbol{\sigma}$. The only possibilities are linear combinations of $\mathbf{n}_\perp\cdot\boldsymbol{\sigma}$ and $\mathbf{n}_{\perp'}\cdot\boldsymbol{\sigma}$. The analog of N is an antihermitian 2x2 matrix that commutes with $\mathbf{n}\cdot\boldsymbol{\sigma}$. The only possibilities are linear combinations of $i\mathbf{1}$ and $i\mathbf{n}\cdot\boldsymbol{\sigma}$.]

To proceed further, specialize to the last option in Equation (5.11), with all v's zero except for $\mathrm{v}_2 = \mathrm{v'}_2 = -1$, i.e.

$$N = \mathbf{n}_\perp\cdot\boldsymbol{\gamma}. \tag{5.12}$$

Then imagine that inside the sphere, the field is as non-singular as it can be, a constant spinor ψ_{in}. Then additionally specify that **k** is small and in Equation (5.10) set $b = (|\mathbf{k}|^2 - (\mathbf{n}\cdot\mathbf{k})^2)^{1/2}$. Now let us consider the situation to first order in b. (I have not figured out how to proceed non-perturbatively; I can only hope that the perturbative conclusion is indicative.) Thus, at the boundary,

$$\psi_{out} \sim (1 + \mathbf{k}\cdot\boldsymbol{\gamma} - (\mathbf{k}\cdot\mathbf{n})\mathbf{n}\cdot\boldsymbol{\gamma})\psi_{in} = \left(\left(1 + \tfrac{2}{3}\mathbf{k}\cdot\boldsymbol{\gamma}\right) + \tfrac{1}{3}\boldsymbol{\gamma}\cdot(\mathbf{k} - \mathbf{3}(\mathbf{k}\cdot\mathbf{n})\mathbf{n})\right)\psi_{in}. \tag{5.13}$$

If the boundary is a sphere, then one can continue these surface values into solutions of the massless Dirac equation outside the sphere:

$$\psi_{out} \sim \left(1 + \tfrac{2}{3}\mathbf{k}\cdot\boldsymbol{\gamma}\right)\psi_{in} + \tfrac{1}{3r^3}R^3\left(\boldsymbol{\gamma}\cdot(\mathbf{k} - \mathbf{3}(\mathbf{k}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\right)\psi_{in}, \tag{5.14}$$

Where R is the radius of the sphere. Then if E is the value of the spinor field far from the sphere (i.e. the externally imposed field), then

$$E \sim \left(1 + \frac{2}{3}\mathbf{k}\cdot\boldsymbol{\gamma}\right)\psi_{in}, \qquad \psi_{in} \sim \left(1 - \frac{2}{3}\mathbf{k}\cdot\boldsymbol{\gamma}\right)E, \tag{5.15}$$

and

$$\psi_{out} \sim E + \tfrac{1}{r^3}\left(\boldsymbol{\gamma}\cdot(\mathbf{k} - \mathbf{3}(\mathbf{k}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\right)\tfrac{R^3}{3}E. \tag{5.16}$$

When one adds all the dipoles induced in this way in all the spheres, the result to first order in **k** will be like Equation (2.3) with $g=(\mathbf{k}\cdot\gamma)/3$, leading to power-law screening (inside the take-away fractal) as in Equation (2.5), with pure imaginary powers that arise as the eigenvalues of a purely antihermitian exponent $-(\mathbf{k}\cdot\gamma)\rho V/\ln\xi$. This construction has obviously exacted a price in rotational non-invariance (explicit appearance of **k**), but presumably, following [1], this has no impact on amplitudes of renormalizable theories in the limit $\varepsilon = 0$.

This construction wouldn't have worked starting from the modified bag ansatz because balancing M with λ or λ_1 or λ_2 in Equation (5.8) or (5.9) makes it impossible to proceed perturbatively and thereby to give preferred treatment to the dipole terms that are linear in **k**.

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