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A Reverse Infinite-Period Bifurcation for the Nonlinear Schrodinger Equation in 2+1 Dimensions with a Parametric Excitation

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Abstract

We consider the nonlinear Schrodinger equation in 2+1 dimensions and an external periodic excitation in parametric resonance with the frequency of a generic mode. Using an adequate perturbation method we get two coupled equations for the amplitude and phase. We show frequency-response curves and demonstrate the existence for the focusing case of a reverse infinite-period bifurcation when the parametric excitation increases its value. The same bifurcation is possible even in the defocusing case but for a different excitation amplitude value.

***Keywords*:** nonlinear Schrodinger equation in 2+1 dimensions; parametric excitation; infinite-period bifurcation; vibration control

1. Introduction

The nonlinear Schrodinger equation, NLSE, is probably the most known and studied nonlinear equation,

$$i\Phi_t + \Phi_{xx} + s|\Phi|^2\Phi = 0, \tag{1}$$

where $\Phi = \Phi(x,t)$is a suitable complex function,$s = \pm 1$ (*s=+1* focusing or *s=-1* defocusing case). Among its physical applications we remember small amplitude gravity waves, Bose-Einstein condensates in highly anisotropic cigar shaped traps or the propagation of light in nonlinear optical fibres and the propagation of Davydov alpha-helix solitons with connected energy transport along molecular chains[1-6]. The case s=+1 is called focusing and there are bright solitons, localized and with spatial fading in the direction of infinity as well as breather solutions. In the other defocusing case, s=-1, we can find dark solitons, with a local spatial dip in amplitude, that is constant at infinity. Moreover, the NLSE could explain the rogue waves occurrence [7]. In the last years many papers performed bifurcations and symmetry breaking analysis for the NLSE [8-15].

A lot of papers consider the 1+1 dimensions case but far fewer papers are devoted to the NLSE in 2+1 dimensions[16]

$$i\Phi_t + \Phi_{xx} + \Phi_{yy} + s|\Phi|^2\Phi = 0 \tag{2}$$

In this paper we consider its nonlinear behavior when there is a parametric resonance

$$i\Phi_t + \Phi_{xx} + \Phi_{yy} + s|\Phi|^2\Phi = f\bar{\Phi}exp(-i\Omega t + 2iK_xX + 2iK_yY) \tag{3}$$

where f is the parametric excitation amplitude, $\bar{\Phi}$is the function Φ complex conjugate
and

$$\omega = K_x^2 + K_y^2 \tag{4}$$

Therefore, the external periodic excitation is in parametric resonance ($\Omega \sim 2\omega$) with the frequency of a generic mode. In Section 2, using an adequate perturbation method [17] we get an approximate solution and two coupled equations for the solution amplitude and phase,

In Section 3, we consider frequency-response curves and demonstrate the possibility of a reverse infinite-period bifurcation. If we increase the parametric excitation from very small values, at the beginning we observe the solution oscillates with its natural frequency but suddenly when the excitation amplitude reaches a critical value the solution begins to oscillate with a very low frequency connected to the external excitation (infinite period or zero frequency bifurcation). If we carefully choose the initial conditions the solution appears immobile, frozen, insensitive to the parametric excitation. For higher excitation values the frequency increases and the system behavior is characterized by a modulated motion. We discuss in some detail the difference between the focusing and defocusing mode showing how even in the last case a reverse infinite-period bifurcation is possible but with different parameters values. In Section 4 we discuss how this paper could be the starting point to understand nonlinear vibrations with various resonances in the NLSE in 2+1 dimensions.

2. Building an approximate solution

In this section we consider a resonance with the external parametric excitation and
scale the forcing term f as $\varepsilon^2 f$ where ε is our bookkeeping device and introduce the slow time

$$\tau = \varepsilon^2 t, \tag{5}$$

because we need to look on larger time scales, in order to detect the amplitude modulation behavior.
The approximate solution $\Phi(x, y, t)$ of equation (3) can be expressed in the following form

$$\Phi(x, y, t) = \varepsilon\Psi(\tau)exp(i\alpha) \tag{6}$$

$$\alpha = K_xX + K_yy - \frac{\Omega}{2}t \tag{7}$$

and the boundary conditions are

$\Phi(0, y, t) = \Phi(L, y, t)\ 0 \leqslant y \leqslant L \quad K_x = \frac{2n_1\pi}{L}$, n_1 integer (8) $\quad \Phi(x, 0, t) = \Phi(x, L, t)\ 0 \leqslant x \leqslant L$

$$K_y = \frac{2n_2\pi}{L},\ n_2 \text{ integer} \tag{9}$$

In order to express the nearness of the excitation frequency Ω to the system frequency ω, we define a detuning parameter σ through the relation

$$\omega_{\underline{n}} = \frac{\Omega}{2} + \varepsilon^2\sigma \tag{10}$$

where

$$\underline{n} = (n_1, n_2) \qquad \omega_{\underline{n}} = \frac{n_1^2\pi^2}{L^2} + \frac{n_2^2\pi^2}{L^2} \tag{11}$$

We have assumed only a mode in the expansion, because we suppose that the system is excited near the natural frequency of a specific linear mode and that mode is not involved in an internal resonance with any other mode. Usually, we can consider that the modes that are not directly excited by an external source or indirectly through an internal resonance will decay with time. Note that the variable change (5) implies that

$$\frac{d}{dt}\left(\Psi \exp\left(-i\frac{\Omega}{2}t\right)\right) = \left(-i\frac{\Omega}{2}\Psi + \varepsilon^2\frac{d\Psi}{d\tau}\right)exp\left(-i\frac{\Omega}{2}t\right). \tag{12}$$

The temporal rescaling (5) allow us to find the asymptotic behavior of the solution, when the nonlinear effects can modify the nonlinear amplitude. The assumed solution (6) is used for the elimination of the predominant linear part of the equation (3) and it allows to calculate the amplitude modulation given by the parametric excitation.

After inserting the assumed solution (6) in the complete equations (3), we obtain the satisfied linear equation

$$\left(\omega_{\underline{n}} - K_x^2 - K_y^2\right)\Psi = 0, \text{(order } \varepsilon^0\text{)}, \tag{13}$$

If we consider equations (3) at the order ε^2 then we find that the function $\Psi(\tau)$ satisfies the nonlinear differential equation

$$i\Psi_\tau - \sigma\Psi + s|\Psi|^2\Psi = f\Psi \tag{14}$$

Expressing the complex-valued function Ψ into its amplitude and phase

$$\Psi = \rho exp(i\theta) \tag{15}$$

we arrive at the model equations

$$\frac{d\rho}{d\tau} + f\rho sin(2\theta) = 0 \tag{16}$$

$$\frac{d\theta}{d\tau} - s\rho^2 + f cos(2\theta) + \sigma = 0 \tag{17}$$

This model system (16-17) describes the amplitude evolution according to our starting assumptions. We observe that the validity of the approximate solution (6) should be expected to be restricted on bounded intervals of the τ-variable and on time-scale $t = O\left(\frac{1}{\varepsilon^2}\right)$, otherwise if we want to find solutions on intervals such that $\tau = O\left(\frac{1}{\varepsilon}\right)$, then the approximate solution (6) loses its validity. In the next section we will find the possibility of a reverse infinite-period bifurcation both for the focusing case(*s=+1*) and the defocusing one*(s=-1*).

3. A reverse infinite-period bifurcation

Periodic solutions of the complete system described by equations (3) correspond to the fixed points of the model system (16-17), which are obtained by the conditions

$$d\rho/dt = d\theta/dt = 0. \tag{17b}$$

Note that the solution exists in the following cases

$$\theta_E = 0 \ , \rho_E = \sqrt{(f+\sigma)},\ f \geqslant -\sigma \text{ (focusing case), } P_{1F} \tag{18}$$

$$\theta_E = 0, \rho_E = \sqrt{(-\sigma - f)} \ , (\sigma + f) \leq 0 \text{ (defocusing case), } P_{1D} \tag{19}$$

$$\theta_E = \frac{\pi}{2} \ , \rho_E = \sqrt{(-f+\sigma)}, \sigma \geq f \text{ (focusing case), } P_{2F} \tag{20}$$

$$\theta_E = \frac{\pi}{2}, \rho_E = \sqrt{(-\sigma + f)}, f \geqslant \sigma \text{ (defocusing case), } P_{2D} \tag{21}$$

We can easily determine their possible stability and through the jacobian matrix we get for the eigenvalues

$$\lambda^2 = -4s\rho_E^2 f cos(2\theta_E) \tag{22}$$

In the focusing case s=+1, P_{1F} is an elliptic point and P_{2F} a saddle point and on the contrary in the defocusing case P_{1D} is a saddle point and P_{2D} an elliptic point. From equation (22) we can get the small oscillations frequency around the elliptic point

$$\Omega = 2\rho_E\sqrt{(f)} \tag{23}$$

Using the trigonometric identity, we arrive at the frequency-response equation

$$\sigma = s_1 f + s\rho_E^2, \tag{24}$$

where ρ_E is now the equilibrium point amplitude, *s=-1* for solutions with $\theta_E = 0$ and moreover*s=+1* for the $\theta_E = \frac{\pi}{2}$solution.(Fig. 1 and Fig. 2)

Around the elliptic point we can get an infinite-period bifurcation but it is more convenient if we consider increasing parametric excitation values so that we can get a reverse infinite-period bifurcation.

We observe the following scenario (focusing case,

For $\sigma < 0$ the excitation frequency is slightly greater than the double of ω, near the equilibrium point P_{1F}). We observe the following scenario:

i) when f=0, no parametric excitation, the wave amplitude is constant

ii) when the excitation is weak, there are no equilibrium points because of (18) and (20), the wave amplitude begins to oscillate with an amplitude proportional to the excitation amplitude *f* and with the natural frequency ω or better

$$\varpi = \omega + \rho^2 \tag{25}$$

iii) if the external excitation increases and reaches the critical value

$$f_C = -\sigma, \tag{26}$$

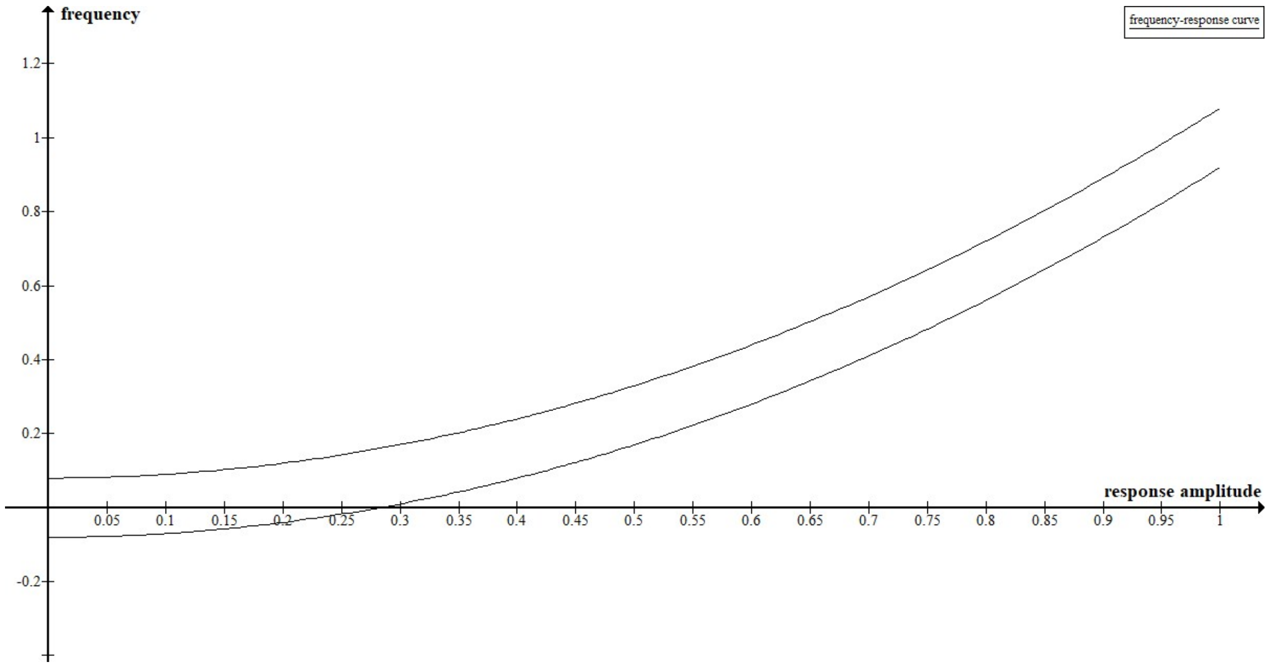

Figure 1: Frequency-response curve for the focusing case(f=0.08). The upper branch correspond to the saddle point and the lower branch to the elliptic point. The infinite period bifurcation is possible when the frequency is negative for the lower branch

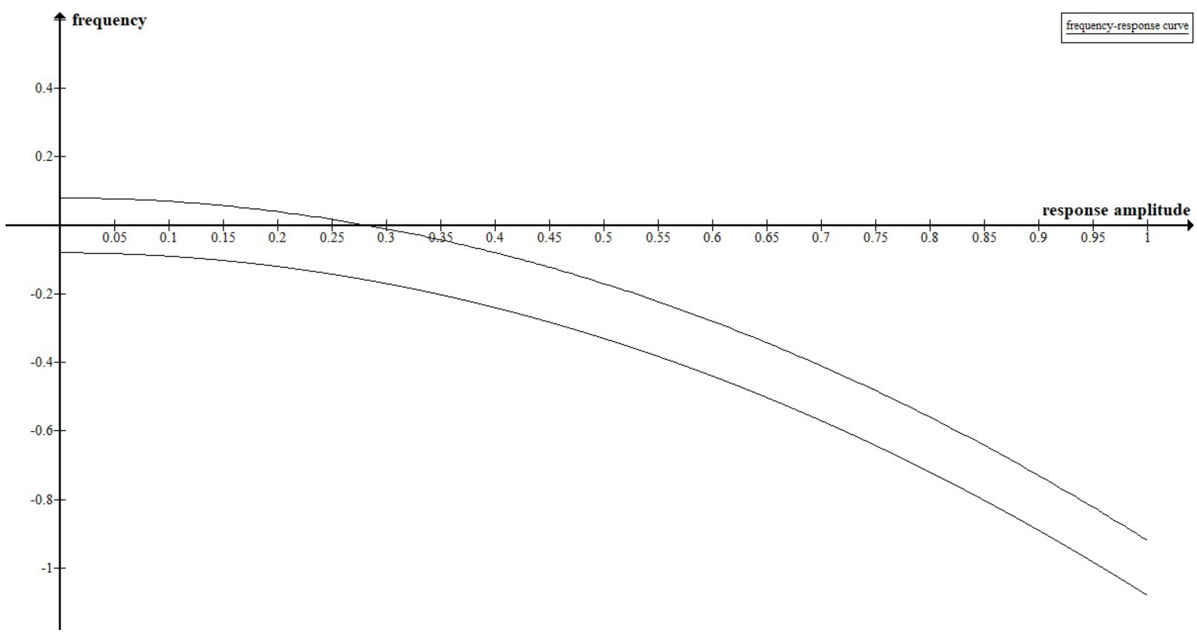

Figure 2: Frequency response curve for the defocusing case (*f*=0.08). The upper branch correspond to the elliptic point and the lower branch to the saddle point. The infinite-period bifurcation is possible when the frequency is positive (upper branch)

the oscillation frequency decreases and the wave seems to collapse but actually it begins a very slow oscillation, dies and is born again, with a large period, a reverse infinite-period bifurcation occurs(Fig. 3).

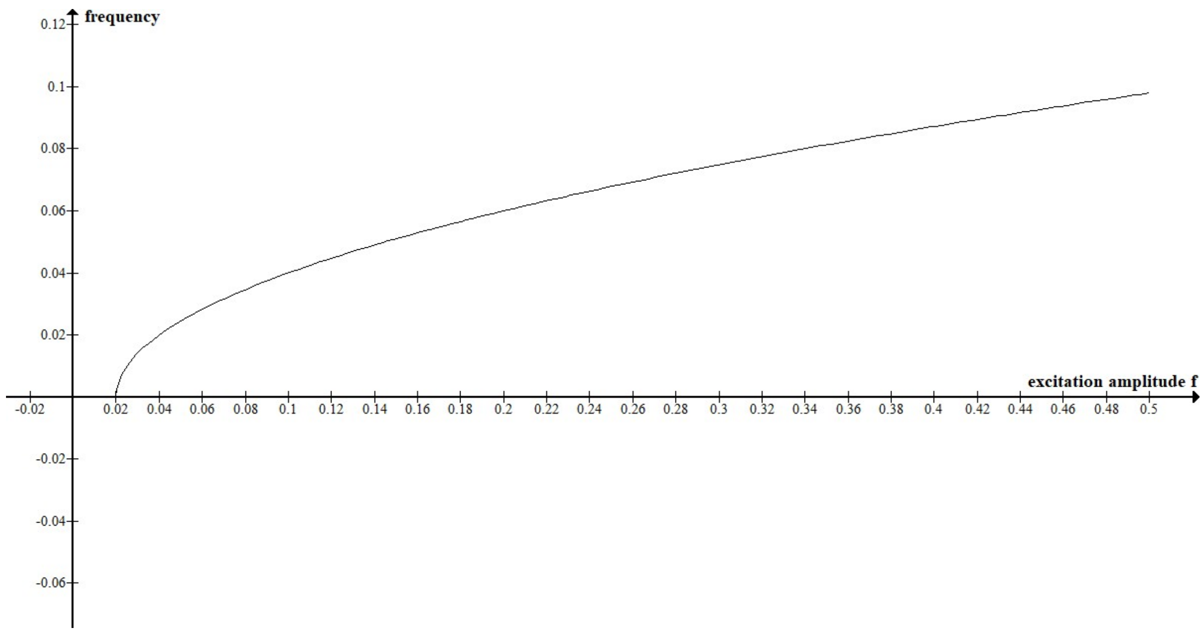

Figure 3: Frequency near the bifurcation point (*f_C=0.02).* The behavior follows a square root law.

We can estimate the oscillation period *T* assuming $\rho \approx \rho_E$ in our calculations

$$T = \frac{2\pi}{\Omega} = \int_0^{2\pi} \frac{(d\theta)}{s\rho_E^2 - \sigma - f\cos(2\theta)} = \frac{\pi\sqrt{(2)}}{\sqrt{(f_C)}\sqrt{(f - f_C)}} \qquad (26)$$

iv) if the initial condition for the wave amplitude is carefully tuned, that is it corresponds to $P_{IF,}$ then the wave amplitude seems immobile, frozen and insensitive to the external excitation with no oscillation.

v) suddenly beyond a critical value $f_C = -\sigma$ and for higher external excitation values , the solution is slowly modulated with a frequency

$$\Omega = \sqrt{\left(f(f + \sigma)\right)} \approx f \qquad (27)$$

We underline the same scenario is possible even for the defocusing case near the point $P_{2D,}$when the excitation frequency Ω is slightly smaller than the natural frequency ω. All the above statements can be easily checked considering the model system (16-17) gets an energy-like function $E(\rho, \theta)$

$$E(\rho, \theta) = \frac{s}{2}\rho^4 - \sigma\rho^2 - f\rho^2\cos(2\theta) \quad \frac{dE}{d\tau} = 0 \qquad (28)$$

We observe that every function in the form $\lambda E(\rho, \theta)$, with λreal number, can be used as energy function. In the focusing case we can take$\lambda=1$, but for the defocusing case the better choice is $\lambda=-1$, so that the elliptic point P_{2D} corresponds to a minimum.

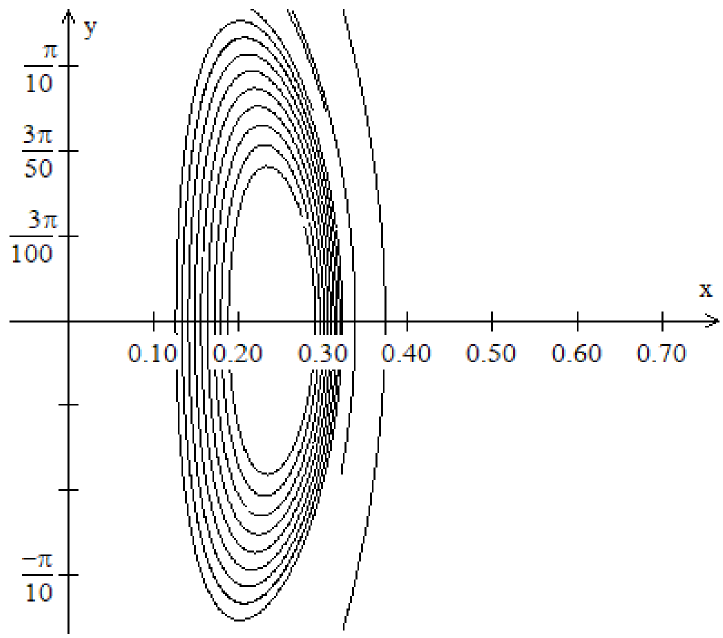

Figure 4: Level curves for the function $E(\rho, \vartheta)$ in the focusing case, $f=0.08$, $\sigma=-0.02$. We can observe the orbits around the elliptic point and the bottleneck account able for the frequency lowering

In Fig. 4, we show the level curves for the energy function (28) near the equilibrium point (18) and we can understand the infinite-period bifurcation is connected to the "bottleneck" we can see in the figure. The fixed point (18) slows the solution motion near it and a lower frequency is excited.

4. Conclusion

We investigated the nonlinear Schrodinger equation in 2+1 dimensions with an external periodic excitation in parametric resonance with a generic mode frequency. We constructed an approximate solution with a suitable perturbation method, previously used for other nonlinear partial differential equations. Two coupled nonlinear equations describe the temporal evolution for the solution amplitude and phase. We show frequency-response curves and demonstrate the existence of a reverse infinite-period bifurcation when the parametric excitation increases its value. The same bifurcation is possible even in the defocusing case but for a different excitation amplitude value. This perturbation method could be applied in order to study other important resonances for the NLSE both in one dimension and two dimensions case.

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