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The Gauge Principle from the Schrodinger-Born Wave Mechanics

P. T. Leung

Department of Physics, Portland State University, P. O. Box 751, Portland, OR 97207
**Corresponding author E-mail*: hopl@pdx.edu

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Abstract

We propose an elementary way of introducing the gauge principle to beginners with a background in only mechanics, electromagnetism, and quantum mechanics. This evolves from an apparent conflict in the Schrodinger-Born formulation of wave mechanics, and does not have to resort to advanced concepts like covariant derivative and minimal coupling. With such an approach, one would have appreciated how interactions can be dictated from consideration of internal symmetry of a physical system, which serves as a principle underlying the foundation of almost all modern physics. In addition, the gauge principle also serves as a resource providing consistency between the Born rule and Schrodinger's wave mechanics.

Keywords**:** non relativistic quantum mechanics, gauge fields, Born rule

1. Introduction

As is well-known, one of the most significant developments in the understanding of the fundamental interactions in the physical universe is the establishment of various gauge theories [1, 2]. These include, for example, the $U(1)$ gauge theory for quantum electrodynamics; the $SU(2)\times U(1)$ electroweak theory; and the $SU(3)\times SU(2)\times U(1)$ standard model. In the introduction of these theories to beginners in a course of elementary particles, the principle of local gauge invariance is often introduced as " a new principle in its own right " through abstract concepts like "covariant derivative" and "minimal coupling" [3]. Here we would like to focus on a more elementary approach by just observing an apparent conflict arising from the Schrodinger-Born formulation of nonrelativistic quantum mechanics.

Historically, the development of gauge theories first started with Weyl in 1918 before quantum mechanics, then followed by Fock in 1926 right after Schrodinger discovered wave mechanics, when Weyl returned in 1929 to complete the establishment of Abelian gauge theory for electrodynamics. The achievement of the modern non Abelian theories started with Klein in

1938† which followed with the Yang-Mill theory in 1954 [1, 2] along with the unpublished work by Shaw in his Ph.D thesis [4]. Moreover, the ingenious observation of Fock immediately after Schrodinger's work was inspired by the contemporary "fifth dimension proposition" for unifying gravity with electromagnetism, from which that additional dimension was proposed by Fock to associate with the wave function of Schrodinger instead [5]. It is interesting to note that Fock's paper was published without being aware of the discovery of the probability interpretation of the wave function by Max Born [6], and hence the statistical interpretation of the wave function was then unknown to him. Here, through a brief review, we point out that had one just examined critically the Schrodinger-Born formulation of wave mechanics, Fock's discovery can immediately be revealed leading at once to the establishment of the fundamental principle of gauge field: symmetry dictates interaction [7].We further point out that besides serving as a mechanism for determining the interaction, this principle also serves as a necessity for ensuring the full consistency between the Born rule and the Schrodinger equation.

2. The Schrodinger-Born Wave Mechanics

As is well-known, in the miraculous year 1926 for Schrodinger, wave mechanics was established through publications of a series of momentous papers – starting from the time independent equation and its application to the hydrogen atom in January to finally the proposition of the time dependent equation in June of that year [8]. In "deriving" the time dependent equation, it is of interest to recall that Schrodinger first proposed an equation containing all real variables with 4^{th} order spatial and 2^{nd} order temporal derivatives. Followed with an ingenious twist, Schrodinger then switched to propose one with 2^{nd} order in space and *first* order in time, with the introduction for the first time the imaginary number $i=\sqrt{-1}$ into a fundamental law of Nature#. This then makes the wave function ψ necessarily complex and thus defies any physical measurement of it. As is well-known, the statistical interpretation was soon proposed by Born in a famous "foot-note" of a paper [6] in which Schrodinger's theory was first applied to scattering problem with:

$$|\psi|^2 = P = \text{probability of finding the particle per spatial volume} \tag{1}$$

and this *real* quantity is the only information extracted from Schrodinger's equation which determines all the observable behavior of the particle.

†*For an interesting discussion on Klein's relevance to the later developments of the non Abelian gauge theory, see the paper by D. J. Gross, "Oscar Klein and Gauge Theory", arxiv.org/abs/hep-th/9411233v .*

And of course, the same had happened with the establishment of matrix mechanics around the same time by Heisenberg, Born and Jordan.

In an introduction to a course in quantum mechanics, most texts often provide a "justification" for the consistency of the Schrodinger-Born formulation of wave mechanics via a derivation of the conservation of probability in the form of a continuity equation. However, there actually exists an *apparent* inconsistency which is not always pointed out and could have led to the discovery of the gauge field principle - symmetry dictates interaction – as elaborated in the following.

3. Apparent conflict in the Schrodinger-Born theory

To motivate one to appreciate the significance of the gauge principle, let us start with the simple case with the time independent Schrodinger equation for a nonrelativistic free electron:

$$\frac{1}{2m}\vec{p}_k^{\,2}\psi = E\psi, \tag{2}$$

where for a free particle, the *kinetic* momentum is also the *canonical* momentum and is given by $\vec{p}_k = \vec{p}_c = -i\hbar\vec{\nabla}$ in the x-representation. To appreciate the apparent conflict between (1) and (2), one simply notices that for a particular solution $\psi(\vec{r})$ satisfying (2), (1) requires that any wave function in the form $\psi_1(\vec{r}) = \psi(\vec{r})e^{i\theta}$ will describe identical physics for the electron in that particular state. However, it is easy to see that $\psi_1(\vec{r})$ will *not* satisfy (2) in general, except for the special case when the phase θ is a constant. For example, when $\theta = \theta(\vec{r})$ is a function of space coordinates, one will have:

$$\vec{p}_k\psi \to \vec{p}_k\psi_1 = -i\hbar\vec{\nabla}\left(\psi e^{i\theta}\right) = \left(-i\hbar\vec{\nabla}\psi + \hbar\psi\vec{\nabla}\theta\right)e^{i\theta} = \left[\vec{p}_k\psi + \hbar\left(\vec{\nabla}\theta\right)\psi\right]e^{i\theta} \tag{3}$$

and

$$p_k^{\,2}\psi \to p_k^{\,2}\psi_1 = \vec{p}_k\left\{\left[\vec{p}_k\psi + \hbar\left(\vec{\nabla}\theta\right)\psi\right]e^{i\theta}\right\} = \left[p_k^{\,2}\psi + 2\hbar\left(\vec{\nabla}\theta\right)\cdot\vec{p}_k\psi - i\hbar^2\nabla^2\theta\psi + \hbar^2\left(\vec{\nabla}\theta\right)^2\psi\right]e^{i\theta} \tag{4}$$

Hence

$$\frac{1}{2m}\vec{p}_k^{\,2}\psi_1 \neq E\psi_1 \tag{5}$$

and we arrive at the conclusion that while ψ and ψ_1 should describe the same physics for the electron according to Born, they do not satisfy the same Schrodinger equation.

4. Restoring phase symmetry

Here we show that in order to restore the gauge (phase) symmetry in wave mechanics, the extra terms $\sim\vec{\nabla}\theta$ in (3) and (4) have to be made physically insignificant. We shall approach this in two steps:

(i) Symmetry dictates interaction

Here we briefly review the fundamental principle of interaction being determined by internal symmetry of the system with reference to the apparent conflict discussed above. To do this, we shall first introduce an external electromagnetic field to interact with the electron via a vector potential $\vec{A}$ in a unique way, while undergoing a simultaneous gauge transformation as the one implied from Maxwell's electrodynamics. We thus propose, under such situation, the kinetic and canonical momenta of the electron no longer identify with each other but generalized to the following relation:

$$\vec{p}_k \to \vec{p}_k = \left(\vec{p}_c - \frac{e}{c}\vec{A}\right) = \left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right) \tag{6}$$

and when $\psi(\vec{r}) \to \psi_1(\vec{r}) = \psi(\vec{r})e^{i\theta(\vec{r})}$, $\vec{A}$ also undergoes simultaneously the usual gauge transformation which ensures Maxwell's fields to be invariant:

$$\vec{A}(\vec{r}) \to \vec{A}_1(\vec{r}) = \vec{A}(\vec{r}) + \left(\hbar c/e\right)\vec{\nabla}\theta(\vec{r}) \tag{7}$$

Note that the factor added to the gradient term is proportional to the fundamental quantum of magnetic flux $\left(hc/2e\right)$ which is to make Eq. (7) dimensionally correct.

With (6) and (7) implemented, we see that (3) and (4) will now transform as follows:

$$\vec{p}_k\psi \to \vec{p}_k\psi_1 = \left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}_1\right)\left(\psi e^{i\theta}\right) \to \left(-i\hbar\vec{\nabla}\psi + \cancel{\hbar\psi\vec{\nabla}\theta} - \frac{e}{c}\vec{A}\psi \cancel{-\hbar\psi\vec{\nabla}\theta}\right)e^{i\theta} = \left(\vec{p}_k\psi\right)e^{i\theta} \tag{8}$$

and

$$p_k{}^2\psi \to p_k{}^2\psi_1 = \vec{p}_k\left(\vec{p}_k\psi_1\right) = \vec{p}_k\left[\left(\vec{p}_k\psi\right)e^{i\theta}\right] = \left(p_k{}^2\psi\right)e^{i\theta} \tag{9}$$

where (8) has been used in the last two steps in deriving (9). Thus the result in (9) will ensure the transformed wavefunciton with an arbitrary spatial dependent phase to satisfy the *same* Schrodinger equation:

$$\frac{1}{2m}\vec{p}_k{}^2\psi_1 = E\psi_1 \tag{9}$$

We hence conclude that to achieve gauge invaiance for *both* the source $\left(\psi \to \psi_1\right)$ and the field $\left(\vec{A} \to \vec{A}_1\right)$, the *interaction* between them has to be determined in a unique way with the kinetic momentum be replaced by one as given in (6). To see this explicitly, one can rewrite (6) in the form:

$$\vec{p}_c = \vec{p}_k + \frac{e}{c}\vec{A} = m\vec{v} + \frac{e}{c}\vec{A} = \frac{\partial L}{\partial \vec{v}} \tag{10}$$

where $L = L\left(\vec{r}, \vec{v}, t\right)$ is the Lagrangian of the charged particle. Integrating (10) with respect to the velocity leads to the following result for the Lagrangian:

$$L = L\left(\vec{r}, \vec{v}, t\right) = \frac{1}{2}mv^2 + \frac{e}{c}\vec{v} \bullet \vec{A} + f\left(\vec{r}\right) = \frac{1}{2}mv^2 + \frac{e}{c}\vec{v} \bullet \vec{A} - e\phi\left(\vec{r}\right) \tag{11}$$

where we have chosen the "integration constant" $f\left(\vec{r}\right)$ to be simply the electrostatic energy for charges not in motion via $L = T - V = 0 - e\phi = -e\phi$. Thus we see that the introduciton of (6) and (7) not only restores full gauge (phase) symmetry so that both the Schrodinger and Maxwell equations remain invariant, but also determines the correct interaction between the source and the field leading to the well-known interaction Lagrangian as shown in the result in (11).

(ii) Full consistency for wave mechanics

Now it is clear that even in the *absence* of any external field, the consistency between the Born rule and Schrodinger's equation can be maintained only if one can implement a principle to make terms $\sim \vec{\nabla}\theta$ to be physically irrelevant when the wavefunciton is allowed to acquire an arbitrary position-dependent phase. Such a principle is exactly the gauge principle as can be seen from the above arguments in (i) by setting $\vec{A}\left(\vec{r}\right) = 0$, still leaving with the possibility of "transforming away" the $\vec{\nabla}\theta$ terms. Hence we conclude that for full consistency in the statistical interpretation of wave mechanics, not only the conservation of probability current (for real potentials) but also the gauge principle have both to be valid --- with the former guarantees the particle's existence and the latter ensures the insignficance of an arbitrary phase factor added to the wave function which may be spatial dependent in general.

5. Conclusion

Without starting from advanced / abstract concepts like covariant derivative, minimum coupling, the Dirac Lagrangian, or models like one with a hypothetical scalar field; the main idea of the gauge principle – symmetry dictates interaction – can be introduced in an elementary way. The key here is to note that to maintain full consistency between Schrodinger's equation and Born's probability interpretation of the wave function which allows one to introduce an arbitrary position-dependent phase factor $\theta\left(\vec{r}\right)$, one has to implement a principle to make terms $\sim \vec{\nabla}\theta$ to be physically irrelevant. Such consistency also ensures expectation values of quantities like the momentum be uniquely defined depsite the arbitrariness in the phase of the wavefunction. The generalization to the time-dependent and relativistic case (including the non Abelian case), and to the derivation of Maxwell's equations with sources can then be done by following standard advanced texts in the usual way [3, 9].

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