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Student Challenges in Understanding Quantum Mechanics: Effect of the "Logic paradigm shift"

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Abstract

For three centuries Newtonian mechanics had been firmly established as a valid theory for the understanding of physical reality; if one understands the laws of physics, then one understands the whole universe. Classical physicists had contented themselves with the search for regularities in measurements and in the physical world. Irregularities were regarded as noises that interfered with the deterministic picture of physical reality. However, from 1900s onwards with the quantum hypothesis, physicists had begun to recognize that the physics of Newton and Maxwell were inadequate for the understanding of all of the physical reality. For example, the interaction of radiation with matter could not be explained from classical physics. This dilemma led to the discovery of quantum mechanics. In this article we explore the challenges that students face in understanding quantum mechanics that arises from paradigm shift in the mode of reasoning about the physical world. The description of physical reality in general and quantum reality in particular requires that we shift our mode of reasoning from classical Boolean logic to quantum non Boolean logic.

***Keywords*:** Quantum logic, Boolean algebra, orthomodular lattices, modal interpretation

1. Introduction

Quantum mechanics not only forms the basis for our understanding of physical phenomena on an atomic and sometimes macroscopic scale but also, it is essential for industrial and technological development. Today, quantum mechanics is applied to most fields of science. However, for students studying quantum mechanics, it is considered an extremely difficult subject. The main difficulty is due to the difference between the conceptual nature of classical physics and quantum mechanics.

Since the early days of quantum mechanics, scientists have been trying to understand the complex and strange quantum behaviors such as time dependence, superposition and measurement problems. This work is trying to show that the challenges facing students in understanding quantum concepts is arising from the logical foundations of quantum mechanics. We argue that the recent developments on the relationship between the geometry of physical reality, set theory and logic may help develop expertise in quantum mechanics.

In physics the description of physical reality is usually based on *definition* and *observation (measurement)*. The process of observation and measurements in physics is associated with logic because logic provides us with basic rules governing the required correlations that we need in order to form propositions about the physical world. In this work we argue that *classical logic* handles the process of observation and measurements in classical physics while *quantum logic* is the study of how logic handles the process of observation and measurements of quantum systems.

George Boole (1854) was the first to give an algebraic formulation to the theory of sets in logic in his work on "An Investigation of the Laws of thought on which are founded the Mathematical Theories of Logic and Probabilities". He established correlation between algebra and Aristotelian propositional logic by showing how logic can be expressed in algebraic terms. In 1937 Birkhoff and von Neumann introduced the idea of quantum logic which not only, treats physical properties of quantum systems as logical propositions, but also studies the relationships and structures formed by these properties, with specific emphasis on quantum measurements. The idea of quantum logic was later on developed, extended and reviewed by Mackey, Jauch, Putnam, Varadarajan, Piron, Beltrametti and Cassinelli, Bub and many others. This work shows that the detailed investigation on the works of Birkhoff and von Neumann provides us with the new insights that might promote understanding of quantum mechanics.

2. Logical Foundations of Classical Mechanics

For three centuries Newtonian mechanics had been firmly established as a valid theory for the description of physical reality. Many physicists had believed that "The state of the system [in classical physics] uniquely determines all phenomena. When the positions and velocities of all objects are known in Newton's mechanics, also the results of all possible measurements are predictable."[1] Due to the firm belief in classical theories, classical physicists had contented themselves with the search for regularities in experiments and in the physical world.

The regularities and order in nature are more appealing to physicists because physics describe properties of physical reality in terms of systems of abstract mathematical objects. Mathematics had become a conceptual model for accessing certain aspects of the physical world. However, according to Robert Giuntini, the mathematical interpretation of a *physical system* (investigated by a physical theory) generally can be represented as a triple that consists of the following ideal objects; a set O, whose elements correspond to the mathematical interpretation of the *physical quantities* concerning the system; a set S, whose elements correspond to the mathematical interpretation of the different *states* that the system may assume; and a *probability – function p*, which determines a probability value for any statement asserting that "the physical quantity A in the state *s* has a value lying in a set *U* of real numbers."[2] We can see clearly from here that there is a close relationship between the *language of sets, logic and probability theory* in Physics which is the main subject that we are going to deal with in this section.

2.1.Propositional Logic

Every measurement of a physical system can always be reduced to a specific kind of measurement called *Yes/No* experiment which corresponds to the observations that permits as an answer only one of the two possible alternatives. The physical quantities that admit two such outcomes are known as *propositions*. "Such yes-no experiments (also called propositions) are taken as the basic entities for building a mathematical formalism."[3]

Propositions are characterized by what we call the *law of the excluded middle*; a proposition is either *true* or *false*, a property is either possessed or not, an event either occurs or not. According to the Aristotelian law of the excluded middle; "the happening of one of them prevents or precludes the happening of the other. In other words, two events are said to be mutually exclusive if they have no sample point in common, i.e. $A \cap B = \phi$.

Classical physics is characterized by such events and propositions. In classical physics, the event that always occurs is a *sure* or *certain* event because the sample space S contains all the sample points and therefore it occurs always. The event that never occurs is the impossible event that contains no sample point of S and is denoted by Ø = { }. The collections of *Yes/No* propositions of a physical system are known as the *logic of the system* and the proposition logic is the area of logic that deals with these propositions.

Operations associated with the two elements are called *binary operations* and propositions that allow only two outcomes such as *Yes/No* outcomes are known as *binary proposition*. Binary propositions about classical systems are measurable or can be experimentally verified and form what we call *binary propositions logic* (bivalent). The *Yes/No* propositions are also the foundations of the digital electric circuits and basis for computer languages. The digital logic gate is the basic building block of all digital electronic circuits and microprocessors. Basic digital logic gates perform logical operations of AND, OR and NOT on binary numbers.

2.2. Isomorphism between finite Boolean algebra, Set Theory and Propositional Logic

It is well known that the syntactic and the semantic aspects of classical propositional logic can be described completely in terms of Boolean algebras [4]. George Boole was the English logician and mathematician who was the first to give an algebraic formulation to the theory of sets in logic. Boole's works have the merit of establishing the close connection between logic and set theory. He established correlation between *algebra and binary propositional logic* by showing how logic can be expressed in algebraic terms.

Although George Boole made significant contributions in a number of areas of mathematics, his two important work that gave decisive impetus to the need to express logical concepts in mathematical form are; *The Mathematical Analysis of Logic, being an Essay towards a Calculus of Deductive Reasoning (1847)* and *An Investigation of the Laws of Thought, on which are Founded the Mathematical Theories of Logic and Probability (1854)*. Boole became the founder of modern symbolic logic through these works. He deduced logic to a propositional calculus, now often known as Boolean algebra based on classical logic. Today, Boolean algebra is important not only in logic, but also in the theory of probability, the theory of lattices and the geometry of sets. Boolean algebra is also fundamental in the development of digital electronics and information theory. Today all standard modern computers perform their functions using two – value Boolean logic.

2.3. Properties of Classical Mechanics fits in with Boolean algebra

We know that a state of an entire system of the classical state is represented by a single point of phase space. The single point of phase space can be represented in terms of Boolean algebra. Therefore, Boolean algebra fits in with the classical state of the system that uniquely determines all phenomena. We may say that properties of a classical mechanical system form a Boolean algebra [5] because using the laws of classical mechanics, the state of the system can be calculated for all times, provided the initial state is known.

In Boolean algebra, every set is supposed to be a subset of a very large set, which is called the universal set for the given set. Thus, any set is a subset of the universal set. A set is said to be finite, if it is empty or contains a finite number of elements, that is, if the number of different elements of the set can be counted and the counting process can be completed. Using geometrical terminology, the universal set is also called *space*, and its elements are called *points*.

In classical mechanics any state of the system at a moment frozen in time is represented as a *point in phase space*; all information about its position or velocity is contained in the coordinates of that point. This means, "given a classical system, a state of maximal knowledge regarding the system is represented by a single point in the phase space associated with the system, or alternatively by an atomic (Dirac) measure concentrated on a single point."[6]

In classical physics, as the system changes in some way, the point moves to a new position in phase space. If the system changes continuously, the point traces a trajectory. A trajectory therefore is the path that an object with mass m in motion follows through space as a function of time t; it is defined by position and momentum simultaneously. In classical physics a trajectory is defined by Hamiltonian mechanics through canonical coordinates. A simple example of a trajectory is the projectile motion.

The *finite number of individual particles* in classical theory and the *finite set of variables* in Boolean algebra make it possible for the properties of a classical mechanical system to fit in a Boolean logic. Algebraically, we can say that the two – valued physical quantity has the values 1 or 0 according as the system *is* or *is not* in the pure state [7]. These two properties that represent classical systems are the ones that also generate *Boolean algebra.* However, this logical model, as we shall see later, is only effective to the description of observable phenomena, which are associated with certainty in their manifestations. According to Bub: "to say that the system has a certain property is to say that the characteristic function P of the corresponding subset takes the *value 1*, which means that the state of the system is represented by a point lying in the subset. To say that the system lacks the property is to say that P takes the *value 0*, which means that the state is represented by a point outside this subset."[5]

Since in classical physics all physical properties i.e. values of physical magnitudes, correspond to mutually exclusive *Yes/No* propositions, they can be represented by orthogonal projection operators in Hilbert space. The fact that the mutually exclusive classical propositions can be represented by orthogonal projection in Hilbert space, classical theories dichotomizes the entities in some given situation of discourse in two mutually exclusive groups; members (those are certainly belong in the set) and non members (those that certainly do not). There is a sharp and unambiguous distinction between these two groups. Wave – particle duality is an example of such distinction that expresses also the inability of the classical concepts "particle" or "wave" to fully describe the behavior of quantum scale objects. The failure of classical physics to explain quantum phenomena such as blackbody radiation, the photoelectric effect, and the hydrogen atom ultimately demolished the logical foundations of classical physics. It can be shown that there is a violation of *joint probability* and *distributive law* by quantum experiments, which suggests the inadequacy of classical logical structure in explaining these experiments.

3. Logical Foundations of Quantum Mechanics

Quantum logic was developed as a model, which is compatible to quantum mechanics features. Birkhoff and von Neumann (1936) were the first to develop the idea that quantum mechanics can be understood best not only in terms of a non – classical calculus but also in terms of non – classical logic such as non – Boolean logic. They had started their theory by discerning the most basic features in quantum mechanics that led to quantum phenomena. They focused on the *relations* between states of quantum systems and then proposed rules that govern such relations that could be considered as a kind of *logic of propositions* about the system. They did this by using the results of the Stern-Gerlach experiment. This led them to establish what is known as the logic of quantum mechanics. And since then a number of works in this line were followed by Jauch, Piron, Varadarajan, Jeffrey Bub and many others.

3.1.Non – Boolean Properties of Quantum Systems

From the logical perspectives we observe that in general, a significant difference between classical and quantum mechanics for the problem of interpretation is that the properties of classical mechanical system form a Boolean algebra while the properties of a quantum mechanical system form a non Boolean algebra. For example classical systems obey a distributive law, which may be stated in two equivalent forms;

$$\begin{cases} a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \end{cases}$$

Quantum systems do not obey a distributive law. The distributive law does not hold in quantum mechanics because the set of all properties detectable by us forms a Boolean algebra while those of quantum mechanics form non Boolean algebra. This is due to the fact that the distributive equivalences of '*and'* and '*or'* does not give the same results for *meet* and *join* in the algebra appropriate to quantum mechanics. According to Bub, "the salient difference between classical and quantum mechanics for the problem of interpretation is that the properties of a classical mechanical system form a Boolean algebra while the properties of a quantum mechanical system form a non – Boolean algebra, not embeddable into a Boolean algebra."[5]

Deep analysis shows that the anomalies in quantum mechanics arise from the complementary variables, which on one hand, serve as the core of the coherent model of reality and on the other hand, seem to violate basic tenets of Boolean logic. Therefore, the non – Boolean algebra, which is the most natural model for quantum logic, seems to be the better candidate for the description of quantum reality.

Therefore, from the logical perspective, the transition from classical mechanics to quantum mechanics means transition from a Boolean to a non – Boolean algebraic structure for the properties of physical system. This implies also the transition from the classical phase space description of the states and dynamical variables of a physical system to the Hilbert space description of quantum mechanics. In the second quarter of the century, Dirac showed that a complex Hilbert space was precisely what is required to form the mathematical basis of quantum mechanics. According to Giuntini, "intuitively, the need to replace phase spaces by Hilbert spaces arises basically from the impossibility to verify position and momentum of a micro particle simultaneously (Heisenberg's uncertainty principle)."[2]

In the usual Hilbert space formulation of quantum mechanics;

- States are *density operators on Hilbert space*,
- Physical quantities are *self – adjoint operators*
- Propositions are *projection operators.*

According to von Neumann's formulation of quantum mechanics, a quantum physical system can be described as triple $\langle O, S, p \rangle$, where O, S, *p* are mathematically identified as follows: O is the set of all self – adjoint operators of the Hilbert space $\mathfrak{H}$ associated with the physical system; S is the set of all density operators of $\mathfrak{H}$; the function *p* is defined by means of Born's rule.[2]

Since projectors on a Hilbert space correspond one – to – one to closed subspaces, they can be expressed in geometrical terms. Therefore, "we denote it by $\wp(\mathcal{H})$and we shall initially refer to it its geometrical description: $\mathcal{H}$is a complex, separable Hilbert space, and $\wp(\mathcal{H})$ is the set of all closed (in the strong topology of $\mathcal{H}$) subspaces of $\mathcal{H}$."[7]

However, one of the types of Hilbert space is of the particular importance to us. This particular elegant type of Hilbert space is one of the infinite dimensions each of which is a complex number. Since it is multi - dimensional, it can handle all the possibilities of a quantum system in one convenient package and in the multi – dimensional Hilbert space, all possible states coexist and add up to the wave function before measurement is performed. Therefore the multi – dimensional and complex Hilbert space, which represents the quantum systems, portrays the physical world more natural than does the classical physics. In this model, the physical world is neither the product of mathematical models nor it is constrained by human mind, but it is characterized by spontaneity and more degrees of freedom. It can be shown that any logically open model of degree n where n is an integer will let a wide range of properties and propositions indeterminate.

The quantum logic pioneered by Birkhoff and von Neumann is derived from Hilbert space quantum mechanics [8]. Thus, all logical primitives, such as propositions, the logical implication relation, as well as the logical operators *and*, or and *not*, must be definable by Hilbert space entities. In order to achieve such a program we need first to identify the logical statements or propositions about a physical system, in particular the elementary one, which can only be *true* or *false*.

John von Neumann noted that *projections on a Hilbert space* can be viewed as *propositions* about physical observables. Therefore, we can treat quantum events (or measurement outcomes) as logical propositions. For example the proposition of the form "*the value of physical quantity A lies in the range B"* which can be represented by a projection operator on a Hilbert space $\mathcal{H}$. According to Karl Svoil, any projection operator on – a Hilbert space corresponds to an elementary proposition. The elementary *true – false* proposition can be spelled out explicitly as: 'The physical system has a property corresponding to the associated closed linear subspace."[8]

We need now to represent logical operations faithfully in terms of Hilbert space operations so that the operations *not, and, or* and the *logical implications* may be identifiable with the classical ones. We achieve this through the partial description of quantum states in orthomodular lattices.

The general idea of quantum logic is based on the isomorphism relation between the set of self – adjoint projection operators defined on a Hilbert space and the set of properties of physical system. The set of all self-adjoint projections operators defined on a Hilbert space in the algebraic terms is the *orthomodular lattice*. "A lattice is orthomodular if and only if it is orthocomplemented and $a \leq b$ implies that $(a, b, a^{\perp})$ is distributive triple. This makes clear that, within orthocomplemented lattices, $Distributivity \Rightarrow Modularity \Rightarrow orthomodularity$."[7]

3.2.Partial Description of a Quantum State in Orthomodular Lattices

We have shown that quantum logics utilize algebraic account of quantum theory. They make use of Boolean algebras, partial Boolean algebras, and orthomodular lattices. These structures can be found embedded in Hilbert spaces, which are the complex topological vector spaces appropriate to quantum mechanics.

In non – Boolean orthomodular description, the two-valued logic, which is inadequate to deal with the complexity of the real world, is considered to be the particular case of the formalism of quantum mechanics. This means that the projectors of two – valued logic are the extreme cases of the quantum effects. This has led to the logic of quantum mechanics, which not only has all values between 0 and 1 but also considers 0 and 1 as limiting cases. According to Michael Dickson, this interpretation "chooses a subset of $\mathcal{L}_x$ to be the domain of the quantum probability measure. Given a judicious choice, it will turn out that the quantum probability measure can be interpreted epistemically, and that at the same time, the properties of everyday objects remain in the domain of the quantum probability measure."[9]

We have already shown in the previous sections that every measurement of a physical system can always be reduced to a specific kind of measurement that permits as an answer only one of the two possible alternatives; that is the outcomes of finite experiment are either *true* or *false*. We have also shown that there is an obvious relationship between finite measurements and finite Boolean sub algebras since the set of its atoms forms a finite measurement. Therefore, the description of a quantum state by using elements corresponding to experimental accessible elements (actually realizable experiments) corresponds to elements of the associated Hilbert lattices. According to Oliver Brunet, orthomodular lattices provide us with a better way to describe the state of quantum systems, based on finite measurements because orthomodular lattices constitute a more general algebraic formalism.

We observe that the description of a quantum state in orthomodular lattices is more expressive way than the orthodox description does. In orthomodular lattices the quantum state (given a feasible measurement) indicates some measurement outcomes, which are known to be impossible with probability = 0 and others with probability that is non – zero, which provides as much information as possible. Therefore, "not only quantum states cannot be regarded as partial states, but partial states are infinitely more informative than quantum states."[10] According to Olivier, this means that a partial description provides information about which outcomes will not occur, and not about which outcomes will occur.

We observe that the description of quantum states in terms of orthomodular description represents the states in quantum mechanics by means of the *space of possible states*. In quantum logic, possible states are considered here as possible states of affairs, which are described by certain logical constructions (indeed conjunctions) or propositions which assign *possible values to physical quantities* [11].

4. Modal Description of Quantum Systems

Quantum logic arises essentially in a probabilistic framework and any probabilistic context gives rise in a natural way to some kind of modalities. Therefore, the dynamics of the modal interpretation provides us with the way forward of overcoming the challenges arising from quantum logical interpretation as a model for the description of quantum reality.

4.1.Van Fraanssen Modal Interpretation

Following Bohm's interpretation of probability, van Fraanssen in 1970s proposed what is known as modal interpretation. The aim of Fraanssen's proposal like Bohm's was to do away with projection postulate which is associated with the collapses of states. The orthodoxy interpretation had proposed the collapse of the wave function or projection postulate in order to account for the process of how the quantum state changes into classical state. However, the problems with the projection postulate are serious; "nobody has (yet) shown how to eliminate 'measurement problem' from the statement of the projection postulate, it may be that there are good reasons nonetheless to suppose that collapse does occur upon measurement."[9]

Modal interpretations have a straightforward way out of the measurement problem because it asserts that, "although only one state of affairs is *actual*, the total state describes all *possibilities* – it gives rise to a probability distribution that comprises both the actual and the possible."[12] All modal interpretations have one common characteristic; they draw a distinction between *physical states* and *theoretical states,* though different authors may use different terms for the same distinction. According to Dickson, "in every modal interpretation, this set depends at least in part on the theoretical state, and may be denoted$\mathcal{A}_w$. The set of possible physical states is a set of maps from $\mathcal{A}_w$ to the set$\{'doesnotoccur',\ 'occurs'\}$, or $\{0, 1\}$. Finally, modal interpretation defines a probability measure over the possible physical states, by adopting the quantum probability measure over$\mathcal{A}_w$."[9]

A physical state is a specification of all possessed properties, for a system (what occurs and what does not occur) and a **theoretical state** determines all probability distributions over possible physical states.

The theoretical state yields probabilistic predictions about which events are now occurring, and which will occur in the future. In other words, if a quantum mechanical state is not equal to an eigenstate, then the value of a given parameter is in *superposition* of all eigenvalues that correspond to the eigenstates that make up the state. A measurement of that parameter will return any of the allowed eigenvalues with probability equal to the square absolute value of the coefficient in front of that eigenstate. In particular, the suggestion is that, granted the usual interpretational links between eigen states of observables and values of physical quantities, a superposition of such eigenstates should be interpreted as representing the joint existence of the corresponding values.[13]

According to all non – collapse interpretation, the state does not only represent what is actually the case in the world we observe, but also contains information about the possibilities that have not become *actual* in our world. Therefore there is a *modal* aspect to all non – collapse approach. Van Fraassen modal interpretation proposes a distinction between what he calls value state of the system and the dynamical state of a system. The *value state* is similar to what we call a *physical state or actual state,* which is conceived as realization in reality. The *dynamic state* is what is considered as *theoretical state or possible value properties,* which may be actualized due to some determinism or indeterminism change of the system. When the dynamical state of a system is mixed, it does not fix the value state, but it determines the set of *possible values states*.

The central idea in van Fraassen's proposal and of modal interpretation in general is that the physical systems at all times posses a number of well – defined physical properties which change in time. This means that the change of a physical system is described also by a change of its properties. The dynamical state determines the set of possible value states and their possible evolution in time.

Moreover, according to van Fraanssen's proposal, a system may have a sharp value of observable even if the dynamical state is not an eigenstate of that same observable. This is a bit different from the traditional eigenstate - eigen value link which says that a system can only have a sharp value of an observable if its quantum state is the corresponding eigenstate. This means the value state corresponds to a given eigenvalue if and only if its dynamical state is an eigenstate of the observable corresponding to that eigenvalue. In quantum systems, not every subset of the state space will correspond to a physical property. "It should be stressed that even in the strictly quantum case, most of the self – adjoint operators actually do not represent 'interestingly' physical quantities; only a few of them represent physical quantities that are useful and meaningful for the description of the physical system (e.g. energy, momentum, position, angular momentum)."[7]

4.2.Van Fraanssen Modal Interpretation in Non – Boolean Structures

From the point of view of the algebraic approach to quantum mechanics, van Fraansen's interpretation is in resonance with non – Boolean algebra, which is the logic structure of quantum propositions. According to Fraansen's, propositions about a physical system cannot be jointly true unless they can be jointly certain according to the standard quantum rules. In other words, propositions about a physical system can be simultaneously true if they are represented by non - commuting observables. This is due to the fact that non – commuting observables impose limits on the possibilities of joint existence of properties themselves, independently of our knowledge. Non – commuting observables do not so much restrict our knowledge about the properties of a system; a physical property and its orthocomplement can both be possible in the same realization.

From the late 1980s, various researchers such as Kochen (1985), Bub and Clifton (1996), Dieks (1995, 2005, 2007), Healey and many others developed further the modal interpretation after realizing the possibilities that it offered to the solution of conceptual problems of quantum mechanics. Quantum logic is now very essential for the development of quantum computers which use the principles of quantum mechanics to deliver huge leaps forward in processing powers.

5. Conclusion

In this paper we have shown that classical mechanics and quantum mechanics are two significantly different paradigms. Therefore, learning quantum mechanics can be challenging even for advanced students who are familiar with the structure of classical mechanics. Many students find it difficult to grasp and visualize the concepts arising from quantum mechanics. In order to help students develop expertise in quantum mechanics, we have developed the logical paradigms that are proper to classical mechanics and quantum mechanics. The familiarization of these paradigms will help students when they encounter a paradigm shift in quantum mechanics in which they must assimilate and accommodate radically complex and strange concepts.

We have shown that from the logical perspective, the transition from classical mechanics to quantum mechanics means transition from a Boolean to a non – Boolean algebraic structure for the properties of physical system. This implies also the transition from the classical phase space description of the states and dynamical variables of a physical system to the Hilbert space description of quantum mechanics. This transition has been done by making use of Boolean algebras, partial Boolean algebras, and orthomodular lattices. These structures are found embedded in Hilbert spaces, which are the complex topological vector spaces appropriate to quantum mechanics.

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