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Expressing the Units of Electricity and Magnetism Directly in the MKS System

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Abstract

Since the unit of electric charge can be chosen independently of the value of the permittivity of free space ε_0, it is shown that all electromagnetic quantities can also be assigned units *directly* in the MKS system. For example, the unit of electric charge can be 1 J as long as ε_0 has units of 1 N. A table is given that makes a comprehensive comparison of the standard units in the Giorgi system with those in two such *direct MKS* schemes. A simple procedure is also described for changing the numerical values of the units in a systematic manner by dividing the various electromagnetic quantities into five distinct classes. This allows one to equate the value of ε_0 to $1/4\pi$, for example, similarly as for the Gaussian system of units, while still retaining the same formulas as in the Giorgi system.

I. Introduction

The fundamental equations of electricity and magnetism are expressed in terms of a rather large set of quantities that are not directly connected to the three units of kinematics, e.g. meter, kilogram and second in the mks system. However, the Giorgi system was introduced [1] in 1901 in order to ensure that whenever *the results of electromagnetic calculations* involve exclusively kinematic quantities, they automatically come out in terms of the mks system of units. By contrast, the purpose of the present study is to design an alternative system of units that allows one to express all electromagnetic quantities directly in the mks system. To see how this goal can be accomplished in a practical way, it is helpful to carry out an extensive review of the Giorgi system of electromagnetic units in order to specify how it is related through experiment and theory to the mks system.

II. Coulomb's Law and the Definition of Electric Charge

The simplest way to begin this analysis is to consider how Coulomb's Law is formulated in the Giorgi system. The force $\mathbf{F_e}$ in Newton (1 N = 1 kg m/s^2) between two electric charges q_i and q_j (expressed in Coul) separated by a distance of $\mathbf{r_{ij}}$ m is given by the vector relation:

$$\mathbf{F_e} = q_i\, q_j\, \mathbf{r_{ij}} / (4\pi\varepsilon_0\, r_{ij}^3) \qquad (1)$$

where ε_0 is the permittivity of free space.
Note that the units for ε_0 are given in such a way ($Coul^2/Nm^2$) so as to insure that the final result is expressed in the mks unit of force (N). The point that needs to be emphasized with regard to this equation is that it serves as a definition of both electric charge and ε_0. In order to satisfy the above requirement in the mks system, it is actually only necessary that the unit for the product of two electric charges q_i q_j divided by ε_0 is Nm^2. This shows that there is an *inherent redundancy* in any system of electromagnetic units *that cannot be removed by experiment.*

We can take advantage of this redundancy for the purpose at hand by defining the unit of electric charge to be some combination of mks units, that is, without introducing a new unit such as the Coulomb for this purpose. We just have to make a corresponding choice of unit for ε_0 to ensure that the force $\mathbf{F_e}$ in eq. (1) is expressed in N. For example, the unit of electric charge could be defined to be the same as for energy (1 J = 1 N m). That would simply mean that the unit of permittivity is N, since then q_i q_j/ε_0 in eq. (1) has the required unit of Nm^2 mentioned above. The very arbitrariness of the above choice of units might tend to make one feel skeptical about such a procedure. What it actually shows, however, is that such quantities are only defined *indirectly* by experiment. As much as we have gotten used to the idea of electric charge over time, it should not be forgotten that there is no other way to determine its magnitude experimentally than to measure the force exerted between it and another charge when they are a certain distance apart.

It is no less permissible to choose a system of electromagnetic units such that ε_0 is dimensionless. This is in fact what is done with the older Gaussian set of units in which charge is expressed in esu. In that system the quantity $4\pi\varepsilon_0$ in Coulomb's Law is missing entirely. One can do this and still remain in the mks system by defining the unit of electric charge to be $N^{0.5}m$. Again, there is no *a priori* reason for avoiding such a choice because charge is only defined experimentally through eq. (1).

There is only one other relationship that must be satisfied in order to extend such an mks-type system to the description of magnetic interactions. The constant μ_0 in the law of Biot and Savart [2] must satisfy the equation from Maxwell's electromagnetic theory:

$$\varepsilon_0 \mu_0 c^2 = 1, \qquad (2)$$

where c is the speed of light in free space (2.99792458×10^8 m/s).
The unit in the Giorgi system is N/Amp^2 or $Ns^2/Coul^2$. If the unit of ε_0 is N, it follows from eq. (2) that the corresponding unit for μ_0 is s^2/Nm^2. Alternatively, if ε_0 is to be dimensionless, then the unit for μ_0 becomes s^2/m^2.

Once the unit of electric charge has been fixed in the mks system, the corresponding units for all other quantities that occur in the theory of electricity and magnetism are determined by the standard equations in which they occur. A fairly extensive list of such quantities illustrating this point is given in Table 1. The corresponding units are always given in terms of those of force, length and time in the mks system. Two sets are given in each case, one in which the unit of charge is Nm and the other in which it is $N^{0.5}m$. The former is referred to as the Nms system so as to distinguish it from the standard mks system for purely kinematic quantities, the other as the $N^{0.5}$ms system, in which ε_0 is dimensionless.

Just a few examples will be given below which emphasize the practicality of the concepts introduced above. The unit of potential (or emf) U is dimensionless in the Nms system since it is proportional to electric charge and inversely proportional to ε_0 and a distance given in m. It has the unit of $N^{0.5}$ in the other system based on the same definition. Since the electric field $\mathbf{E}$ is the gradient of a potential, it follows that it has a unit of m^{-1} in the Nms system and $N^{0.5}/m$ in the other. The unit of current I is Nm/s in the former case, while that of resistance R (I=V/R) is accordingly s/Nm. In the $N^{0.5}$ms system, R has the unit of s/m, i.e. the reciprocal of that of velocity, whereas the unit for I is $N^{0.5}m/s$.

In the Giorgi system of units, the magnetic force $\mathbf{F_m}$ for a given charge q moving with velocity **v** in magnetic field **B** is defined as:

$$\mathbf{F_m} = q\,\mathbf{v} \times \mathbf{B}. \qquad (3)$$

It therefore follows that **B** has the unit of s/m^2 in the Nms system and $N^{0.5}s/m^2$ in the $N^{0.5}$ms system. The Nms unit of magnetic flux (Weber in the Giorgi system or Tesla m^2) is s, consistent with the requirement that an induced emf, which is dimensionless in the Nms system of units, is given by the derivative of the magnetic flux with respect to time. In the $N^{0.5}$ms system its unit is $N^{0.5}$s. It is easy to show that the units are consistent for Maxwell's equations in both of these systems of units. For example, the differential form of Faraday's law of electromagnetic induction given by Eq. (4) has the units of m^{-2} on both sides in the Nms system and $N^{0.5}/m^2$ in the other.

$$\mathbf{curl\ E} = - \partial \mathbf{B}/\partial t, \qquad (4)$$

III. A Simple Scaling Procedure for Electromagnetic Quantities

The interdependency of the definitions of electric charge q and permittivity ε_0 also presents other options for the choice of units for electromagnetic quantities than those of the Giorgi system. The esu system of units [3] employs a much smaller unit of electric charge than Coul, for example, which therefore makes it unnecessary to include the $4\pi\varepsilon_0$ factor in eq. (1), which is to say that in this system of units, $\varepsilon_0 = 1/4\pi$. The system of atomic units, in which the electronic charge e serves as the unit of electric charge, makes the same choice for ε_0. In the present section we will illustrate how the various electromagnetic units of the Giorgi system can be modified in a systematic manner so that the latter condition is also fulfilled for mks units.

To begin this discussion it is important to note that the value of ε_0 in the Giorgi system is based directly on the speed of light in mks units: the value of $4\pi\varepsilon_0$ is equal to $10^7/c^2$. Since the speed of light in free space is no longer measured but is simply defined by international convention to have the above value [4], it follows that there is also no need to determine quantities such as the Coulomb (Coul) and ε_0 that are ultimately based on the value of c. A convenient quantity with which to scale the various standard Giorgi units is,

$$K = (4\pi\varepsilon_0)^{-0.5} = 10^{-3.5}c \approx 94802.$$

In the following we will refer to the new set of units as the KNms system. First, we define the corresponding value of the permittivity as $\varepsilon_0' = K^2\varepsilon_0$, so that $4\pi\,\varepsilon_0' = 1$ N. In general, the units in the new system are those given in Table 1 under the Nms heading, that is, with the unit of electric charge equal to 1 J = 1 Nm. It should be clear, however, that the numerical value attached to ε_0' in the new system is completely independent of this choice. One could just as well choose the unit of charge to be $N^{0.5}$m, for example, or any other combination of N, m and s, as long as one adheres to the requirements already discussed in Sect. II.

The objective in changing the numerical values of electromagnetic constants such as ε_0 is clearly to simplify computations in this important area of physics. One of the problems with changing over from the Giorgi to the Gaussian system of units is that in many cases this requires using different formulas for the same interaction. One can avoid this difficulty by agreeing at the outset that all formulas in the new KNms system will be the same as for the Giorgi system, since the latter have become standard over the past century. Let us consider eq. (1) as the first example. In order to retain the same form for this equation while using the above value for ε_0', it is simply necessary to change the numerical value of each electric charge. Specifically, one has to change the unit of charge to K^{-1} Coul. This means that the value of the electronic charge (e') becomes K times larger than the standard value in Coul, i.e, $e' = 94802 \times 1.602 \times 10^{-19}$ J $= 1.5187 \times 10^{-14}$ J. In effect then, the change from the Giorgi to the KNms system of units occurs by multiplying both the numerator and denominator in eq. (1) by the same factor (K^2). The result is that one has the same form for eq. (1) as in the Gaussian or atomic unit versions, i.e, where $4\pi\varepsilon_0 = 1$ and thus does not appear explicitly.

The main point that the above discussion reveals is that it is useful to divide the variables that commonly occur in the theory of electricity and magnetism into classes according to the way in which their numerical values need to be scaled. In the KNms system, this means that each such variable needs to be associated with a specific power of K. This information has also been given in Table 1 in each case. Since $\varepsilon_0' = K^2\varepsilon_0$, for example, it is necessary to multiply the Giorgi value for μ_0 by K^{-2} in order to be consistent with eq. (2), that is, without changing the value of c. As a result, $\mu_0' = 4\pi/c^2$. Again, the preferred approach is not to eliminate ε_0' and μ_0' from the formulas in the KNms system, rather only to change their numerical values relative to those in the Giorgi mks system so that the form of the standard equations in the latter system is completely retained.

Other quantities that belong to the same K-class in Table 1 as electric charge are charge densities ρ and σ, dipole moment **μ**, quadrupole moment Q, current I, current density **J**, magnetic dipole moment **m**, magnetization **M** and magnetic intensity **H**. The corresponding quantities of K^{-1} type are: electric potential U, electric field **E**, magnetic field (or induction) **B**, magnetic flux Φ and magnetic vector potential **A**. A check of all formulas in which the latter quantities appear shows that they always occur with counterparts in the K class mentioned first, as, for example, q and **B** in eq. (3) or q and **E** in the corresponding expression for electric force.

Table 1. Correlation of the units of electromagnetic quantities in various systems. The standard Giorgi system is compared with two alternatives, the Nms and $N^{0.5}$ms systems, whose units are exclusively multiples of N, m and s in the standard mks system for strictly mechanical variables. The quantities are also subdivided into K-type scaling classes, as discussed in Sect. III.

Quantity	**Symbol**	**Giorgi**	**Nms**	**$N^{0.5}$ms**	**Scaling class**
Electric charge	q	Coul	Nm	$N^{0.5}$m	K
Permittivity	ε or ε_0	$Coul^2/Nm^2$	N	____	K^2
Current/mmf	I	Amp	Nm/s	$N^{0.5}$m/s	K
Permeability	μ or μ_0	N/Amp^2	s^2/Nm^2	s^2/m^2	K^{-2}
Potential/emf	V	Volt	_____	$N^{0.5}$	K^{-1}
Resistance/impedance	R/Z	Ohm	s/Nm	s/m	K^{-2}
Electric field	**E**	Volt/m	1/m	$N^{0.5}$/m	K^{-1}
Volume charge density	ρ	$Coul/m^3$	N/m^2	$N^{0.5}/m^2$	K
Surface charge density	σ	$Coul/m^2$	N/m	$N^{0.5}$/m	K
Electric dipole moment	$\boldsymbol{\mu}_e$	mCoul	Nm^2	$N^{0.5}m^2$	K
Electric quadrupole moment	Q_{ij}	m^2 Coul	Nm^3	$N^{0.5}m^3$	K
Electric polarization	**P**	$Coul/m^2$	N/m	$N^{0.5}$/m	K
Electric displacement	**D**	$Coul/m^2$	N/m	$N^{0.5}$/m	K
Electric susceptibility	χ	Coul/mVolt	N	____	K^2
Polarizability	α	m^2Coul/Volt	Nm^3	m^3	K^2
Coefficient of potential	p_{ij}	Volt/Coul	1/Nm	1/m	K^{-2}
Capacitance/coeff. of capacitance	C orc_{ij}	Coul/Volt	Nm	m	K^2
Current density	**J**	$Coul/m^2s$	N/ms	$N^{0.5}$/ms	K

Conductivity	g	Coul/msVolt	N/s	1/s	K^2
Resistivity	η	msVolt/Coul	s/N	s	K^{-2}
Magnetic flux	Φ	Weber	s	$N^{0.5}s$	K^{-1}
Magnetic induction	**B**	Weber/m^2	s/m^2	$N^{0.5}s/m^2$	K^{-1}
Magnetic vector potential	**A**	Weber/m	s/m	$N^{0.5}s/m$	K^{-1}
Magnetic scalar potential	U*	Amp	N/ms	$N^{0.5}m/s$	K
Magnetic dipole moment	**M**	m^2Amp	Nm^3s	$N^{0.5}m^3/s$	K
Magnetization	**M**	Amp/m	N/s	$N^{0.5}/s$	K
Inductance	L	Henry	s^2/Nm	s^2/m	K^{-2}
Magnetic current per unit area	$\mathbf{J_m}$	Amp/m^2	N/ms	$N^{0.5}$/ms	K
Magnetic intensity	**H**	Amp/m	N/s	$N^{0.5}/s$	K
Reluctance	*R*	Amp/Weber	Nm/s^2	m/s^2	K^2
Admittance	Y	Mho	Nm/s	m/s	K^2

Some quantities do no not have to be scaled at all (K^0-type). They include all dimensionless quantities such as magnetic susceptibilities and refractive indices. The same is of course true for all non-electromagnetic quantities such as force, energy and angular momentum. A less trivial example is the Poynting vector (**E × H**), which is a product of a K^{-1}- and K-type variable, respectively. All other commonly occurring quantities are either of K^2- or K^{-2}-type. In addition to ε_0 among the former are the dielectric constant ε and electrical susceptibility χ (Table 1), as well as polarizability, capacitance, reluctance, conductivity and admittance. Some examples of K^{-2}-type are in addition to μ_0: permeability μ, resistance, coefficient of potential p_{ij}, resistivity η and inductance L. The latter quantity is defined as dΦ/dI, which is a ratio of a K^{-1}–type quantity to the current, which is of K-type.

The conversion factors between the Giorgi and the present KNms systems of electromagnetic units for a number of the most commonly used quantities are given in Table 2. Unlike the case for the corresponding conversion between the Gaussian and Giorgi systems [3], the formulas in which they are to be used respectively are exactly the same, as discussed above. To be specific, we have given these factors as functions of c rather than of K itself. Clearly, any other value of K could be used while still allowing the Giorgi formulas to be retained in the new system of units. The value of the electric charge in any such system of units is K times that of the numerical value in the Giorgi system ($e=1.602\times10^{-19}$). As long as one adheres to the scheme of dividing the variables into K-type classes according to the prescriptions of Table 1, this information is sufficient to characterize any new system of this type. In other words, the scaling procedure is always perfectly defined by the value chosen for K in a specific instance.

Table 2. Conversion of various electromagnetic units from the Giorgi to the KNms system discussed in Sect. III (c is the speed of light in free space, 2.99792458×10^8 m/s).

Quantity	**Giorgi**	**KNms**
Electric charge	1 Coul	$10^{-3.5}$c Nm
Electric current	1 Amp	$10^{-3.5}$c Nm/s

$4\pi\varepsilon_0$	10^7c^{-2} Coul^2/Nm^2	1 N
$\mu_0/4\pi$	10^{-7} N/Amp^2	c^{-2} s^2/Nm^2
Electric field	1 Volt/m	$10^{3.5}$ c^{-1} 1/m
Potential	1 Volt	$10^{3.5}$ c^{-1}
Magnetic induction	1 Weber/m^2	$10^{3.5}$ c^{-1} s/m^2
Magnetic intensity	1 Amp/m	$10^{-3.5}$c N/s
Magnetic flux	1 Weber	$10^{3.5}$ c^{-1} s
Electric displacement/polarization	1 Coul/m^2	$10^{-3.5}$c N/m
Capacitance	1 Farad=Coul/Volt	$10^{-7}c^2$ Nm
Inductance	1 Henry	10^7c^{-2} s^2/Nm

IV. Conclusion

Over time we have come to believe that we experience electric charge directly, but the fact is that this is only a theoretical quantity that allows us to calculate the Coulomb force between particles at any distance. Force is something very tangible whereas charge is not, despite what our intuition and training tell us. The determination of a **B** field also can only be accomplished by measuring the force between electric charges in relative motion. All the other myriad quantities mentioned in Table 1 have definitions that in one way or another involve explicit measurements of such forces. As long as one remains consistent, that is, adheres to the simple restrictions outlined in Sect. II, one system of electromagnetic units will inevitably lead to the same set of measured values for force, distance and elapsed time as any other.

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