Quasi-exactly solvable PT-symmetric sextic oscillators resulting from real quotient polynomials

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(Received 16 January 2019, Accepted 16 March 2019, Published 23 March 2019)

Abstract

We present a method of constructing PT-symmetric sextic oscillators using quotient polynomials and show that the reality of the energy spectrum of the oscillators is directly related to the PT symmetry of the respective quotient polynomials. We then apply the method to derive sextic oscillators from real quotient polynomials and demonstrate that the set of resulting oscillators comprises a quasi-exactly solvable system that contains the real, quasi-exactly solvable sextic oscillator. In this framework, the classification of the PT-symmetric sextic oscillators on the basis of whether they result from real or complex quotient polynomials is a natural consequence.

Keywords: PT symmetry, sextic oscillators, quasi-exact solvability, quotient polynomials

1. Introduction

In PT-symmetric quantum mechanics, the "traditional" Hermitian Hamiltonians are replaced by PT-symmetric Hamiltonians, i.e. by Hamiltonians that are invariant (unchanged) under the combined action of parity (space reflection) and time reversal[1] [1]. PT symmetry in quantum mechanics was first proposed by Bender and Boettcher in 1998 [2], with the introduction of a class of non-Hermitian, PT-symmetric Hamiltonians with real spectra, and rapidly became an active area

[1] Parity and time reversal are two important discrete transformations, which are represented by the operators $\hat{P}$ and $\hat{T}$, respectively. By definition, the parity operator changes the sign of the position operator, and also, it changes the sign of the momentum operator (to understand why, you may think classically). Then, as it leaves the position-momentum commutator unchanged, it must also leave unchanged the imaginary unit, since $\left[\hat{x}, \hat{p}\right] = i\hbar$, and as a result, the parity operator is *linear*. On the other hand, the time-reversal operator leaves the position operator unchanged (time reversal is independent of space reflection), but it changes the sign of the momentum operator, as it changes the sign of time. Thus, the time-reversal operator changes the sign of the position-momentum commutator, and as a result, it must also change the sign of the imaginary unit, which means that it is an *antilinear* operator.

of research in theoretical physics, with more than two thousand papers already published and many international conferences devoted to the subject [3]. Also, in 2007, El-Ganainy et al. [4] demonstrated the connection of PT-symmetric quantum mechanics to optics and the first optical experiments were conducted, which were followed by many more, in such diverse areas of applied physics as optical wave guides, lasers, microwave cavities, superconducting wires, graphene, and metamaterials [5-8].

PT-symmetric Hamiltonians can have the two fundamental properties that any consistent quantum theory possesses: real energy eigenvalues and unitary time evolution (probability conservation) [1]. In quantum mechanics, complex potentials generally model open, i.e. non-isolated, systems that exchange energy with their environment. Particularly, a complex potential with positive imaginary part describes a system that absorbs energy from its environment, while a complex potential with negative imaginary part describes a system that releases energy to its environment. However, a purely imaginary and antisymmetric potential, which is thus PT-symmetric, models a balanced distribution of sources and sinks of energy in space [9], and then a complex PT-symmetric potential models a system interacting with its environment in such a way that its energy loss and gain are balanced.

The introduction of PT-symmetric Hamiltonians was followed by the introduction of new quasi-exactly solvable PT-symmetric potentials. In this framework, PT-symmetric sextic potentials were introduced and studied [10, 11]. The purpose of the present paper is to use the quotient-polynomial approach we presented in [12] to construct PT-symmetric sextic oscillators and exploit the option to derive complex oscillators from real quotient polynomials. In addition, we wish to demonstrate that the set of PT-symmetric sextic oscillators resulting from real quotient polynomials comprises a quasi-exactly solvable system that contains, as a special case, the real, quasi-exactly solvable sextic oscillator, and also to highlight a new classification of the PT-symmetric sextic oscillators on the basis of whether they come from real or complex quotient polynomials.

The rest of the paper is organized as follows: in the next section, making an ansatz for the wave function, we introduce the quotient polynomial and use it to construct PT-symmetric sextic oscillators, demonstrating that the PT invariance of the quotient polynomial is a necessary and sufficient condition for the reality of the energy spectrum of the respective oscillator. In section 3, taking advantage of the option provided by our approach, we construct complex oscillators from real quotient polynomials. We specifically examine the cases where the non-negative integer parameter of the potential takes the values 0, 1, 2, and 3, and show that the set of resulting oscillators contains the real, quasi-exactly solvable sextic oscillator and becomes richer as the parameter increases, which signifies the quasi-exact solvability of the system. Finally, in section 4, we summarize and conclude.

2. Construction of PT-symmetric sextic oscillators from quotient polynomials

In line with the analysis presented in [12], we choose a length scale l and do the transformations $x \to lx$, $E \to \hbar^2 E/2ml^2$, and $V(x) \to \hbar^2 V(x)/2ml^2$. Then, the position x, the energy E, and the potential $V(x)$ become dimensionless and the stationary Schrödinger equation for our system reads

$$\psi''(x) + (E - V(x))\psi(x) = 0,$$

where $\psi(x)$ is an energy eigenfunction of the system.

Next, we establish our ansatz scheme by seeking eigenfunctions of the form

$$\psi(x) = A_n p_n(x) \exp(g_4(x)) \tag{1}$$

where A_n is the normalization constant, $p_n(x)$ is an n-degree polynomial, and $g_4(x)$ is a fourth-degree polynomial with negative leading coefficient so that (1) is square-integrable in $\mathbb{R}$ [2]. Since x is dimensionless, the dimensions of $p_n(x)$ is carried by its coefficients. Thus, incorporating the leading coefficient of $p_n(x)$ into the normalization constant A_n, we make $p_n(x)$ both monic and dimensionless. As exponent, the polynomial $g_4(x)$ must be dimensionless too, and since x is dimensionless, the coefficients of $g_4(x)$ are also dimensionless. The constant term of $g_4(x)$ is a multiplicative constant to (1), thus it can also be incorporated into the normalization constant A_n. Finally, choosing the length scale l appropriately, we can set the leading coefficient of $g_4(x)$ to a desirable negative value and without loss of generality, we write $g_4(x)$ as

$$g_4(x) = -\frac{1}{4}x^4 + \frac{g_3}{3}x^3 + \frac{g_2}{2}x^2 + g_1 x \tag{2}$$

If the polynomial (2) is not PT-symmetric, the eigenfunction (1) cannot be either even or odd under PT-symmetry, and, in this case, the PT symmetry is broken [1]. Since we wish to explore the possibility that the PT symmetry remains unbroken, we demand that the polynomial (2) is PT-symmetric, which means that the coefficients g_1 and g_3 are imaginary, while g_2 is real.

Plugging the ansatz eigenfunction (1) into the stationary Schrödinger equation and solving for the potential yields

$$V(x) = \frac{p_n''(x) + 2g_4'(x)p_n'(x)}{p_n(x)} + g_4'^2(x) + g_4''(x) + E$$

[2] Generally, the stationary Schrödinger equation for PT-symmetric potentials is solved along a properly chosen contour on the complex plane [1]. However, for the potential we examine, it suffices to solve the equation on the real axis.

In the last equation, the potential and the expression $g_4'^2(x)+g_4''(x)+E$ are polynomials, thus their difference is also a polynomial, and then the expression $\left(p_n''(x)+2g_4'(x)p_n'(x)\right)/p_n(x)$ is a polynomial too. Moreover, this polynomial is quadratic, since

$$\deg\left(\left(p_n''+2g_4'p_n'\right)/p_n\right)=\deg\left(p_n''+2g_4'p_n'\right)-\deg(p_n)=\deg\left(g_4'p_n'\right)-\deg(p_n)$$
$$=\deg\left(g_4'\right)+\deg\left(p_n'\right)-\deg(p_n)=3+n-1-n=2$$

Thus, we can write

$$p_n''(x)+2g_4'(x)p_n'(x)=-q_2(x;n)p_n(x) \tag{3}$$

where

$$q_2(x;n)=q_2(n)x^2+q_1(n)x+q_0(n) \tag{4}$$

We'll refer to $q_2(x;n)$ as the quotient polynomial. The minus sign on the right-hand side of (3) is put in for convenience.

Using (2) and (4), (3) is written as

$$p_n''(x)+2\left(-x^3+g_3x^2+g_2x+g_1\right)p_n'(x)=-\left(q_2(n)x^2+q_1(n)x+q_0(n)\right)p_n(x) \tag{5}$$

Since $p_n(x)$ is monic, the highest-order terms on the left and right hand sides of (5) are, respectively, $-2nx^{n+2}$ and $-q_2(n)x^{n+2}$, thus

$$q_2(n)=2n,$$

and then the quotient polynomial (4) and the differential equation (5) are respectively written as

$$q_2(x;n)=2nx^2+q_1(n)x+q_0(n) \tag{6}$$

$$p_n''(x)+2\left(-x^3+g_3x^2+g_2x+g_1\right)p_n'(x)=-\left(2nx^2+q_1(n)x+q_0(n)\right)p_n(x) \tag{7}$$

In terms of the quotient polynomial, the potential is written as

$$V(x)=-q_2(x;n)+g_4'^2(x)+g_4''(x)+E \tag{8}$$

Since the polynomial $g_4(x)$ is assumed PT-symmetric, its first derivative is odd, i.e. it changes sign, under PT symmetry, while its second derivative is PT-symmetric, as it is easily seen by (2). Therefore, the polynomial $g_4'^2(x)+g_4''(x)$ is also PT-symmetric, and since we want the potential to also be PT-symmetric, from (8) we derive that the polynomial $q_2(x;n)-E$ must be PT-symmetric too. But, using (6), we have

$$q_2(x;n)-E=2nx^2+q_1(n)x+q_0(n)-E,$$

and the action of a PT transformation on $q_2(x;n)-E$ then yields

$$2nx^2 - q_1^*(n)x + (q_0(n)-E)^*,$$

where the asterisk denotes complex conjugation. For the polynomial $q_2(x;n)-E$ to be invariant under the PT transformation, $q_1(n)$ must be imaginary, while $q_0(n)-E$ must be real. Then, if the energy is real $q_0(n)$ is also real and the quotient polynomial is then PT-symmetric, and conversely, if the quotient polynomial is PT-symmetric, $q_0(n)$ is real, and then the energy is real too. Therefore, *the quotient polynomial is PT-symmetric if and only if the energy is real*.

The potential (8) is expressed up to an additive constant and to determine it uniquely, we choose its value at zero to be zero, i.e. $V(0)=0$, which, by means of (2), (6), and (8) reads

$$E = q_0(n) - (g_1^2 + g_2) \tag{9}$$

Since g_1 is imaginary and g_2 is real, the expression $g_1^2 + g_2$ is real, as is the constant term $q_0(n)$ of a PT-symmetric quotient polynomial, and then (9) gives real energies, as it should. Then, using (2) to express the first and second derivatives of $g_4(x)$ and substituting into (8) along with (6) and (9), we finally end up to

$$\begin{aligned} V(x) &= x^6 - 2g_3x^5 + (g_3^2 - 2g_2)x^4 + 2(g_2g_3 - g_1)x^3 + (g_2^2 + 2g_1g_3 - (2n+3))x^2 \\ &+ (2g_3 + 2g_1g_2 - q_1(n))x \end{aligned} \tag{10}$$

Since g_1 and g_3 are imaginary, g_2 is real, and $q_1(n)$ is imaginary, the couplings $g_3^2 - 2g_2$ and $g_2^2 + 2g_1g_3 - (2n+3)$ are real, while the couplings $g_2g_3 - g_1$ and $2g_3 + 2g_1g_2 - q_1(n)$ are imaginary, and then the potential (10) is a complex PT-symmetric sextic oscillator.

3. Complex oscillators from real quotient polynomials

The PT-symmetric quotient polynomial (6) is real if and only if $q_1(n)$ vanishes. In this case, the differential equation (7) and the potential (10) become, respectively,

$$p_n''(x) + 2(-x^3 + g_3x^2 + g_2x + g_1)p_n'(x) = -(2nx^2 + q_0(n))p_n(x) \tag{11}$$

$$\begin{aligned} V(x) &= x^6 - 2g_3x^5 + (g_3^2 - 2g_2)x^4 + 2(g_2g_3 - g_1)x^3 + (g_2^2 + 2g_1g_3 - (2n+3))x^2 \\ &+ 2(g_3 + g_1g_2)x \end{aligned} \tag{12}$$

We'll construct PT-symmetric sextic oscillators from real quotient polynomials in the cases where $n = 0,1,2,3$.

3.1 The case n=0

The polynomial $p_0(x)$ is monic and of degree 0, thus it equals 1, and then from (11), we obtain that $q_0(n)$ vanishes, and then the quotient polynomial vanishes in this case. Then, the wave function (1), the energy (9), and the potential (12) take the form, respectively,

$$\psi(x) = A\exp\left(-\frac{1}{4}x^4 + \frac{g_3}{3}x^3 + \frac{g_2}{2}x^2 + g_1 x\right) \tag{13}$$

$$E = -\left(g_1^{\ 2} + g_2\right) \tag{14}$$

$$\begin{aligned} V(x) = {} & x^6 - 2g_3x^5 + \left(g_3^{\ 2} - 2g_2\right)x^4 + 2\left(g_2g_3 - g_1\right)x^3 + \left(g_2^{\ 2} + 2g_1g_3 - 3\right)x^2 \\ & +2\left(g_3 + g_1g_2\right)x \end{aligned} \tag{15}$$

The wave function (13) is energy eigenfunction of the PT-symmetric sextic oscillator (15), with the real energy (14). If g_1 and g_3 vanish, (13) – (15) give, respectively, the ground-state wave function and energy of the real, quasi-exactly solvable sextic oscillator for $n = 0$, in line with [12].

3.2 The case n=1

The polynomial $p_1(x)$ is monic and of degree 1, thus it has the form

$$p_1(x) = x + p_0 ,$$

and then (11) reads

$$2\left(-x^3 + g_3x^2 + g_2x + g_1\right) = -\left(2x^2 + q_0(1)\right)\left(x + p_0\right)$$

The coefficients of the same-degree terms in x on both sides of the last equation must be equal, and thus

$$p_0 = -g_3 \tag{16}$$

$$q_0(1) = -2g_2 \tag{17}$$

$$-q_0(1)\,p_0 = 2g_1$$

Substituting (16) and (17) into the last equation yields the condition

$$g_1 = -g_2g_3 \tag{18}$$

for the quotient polynomial to be real in the case $n = 1$.

If g_3 vanishes, from (18) we derive that g_1 vanishes too, while from (16) we see that p_0 also vanishes. Then, the wave function (1) reads

$$\psi(x) = Ax\exp\left(-\frac{1}{4}x^4 + \frac{g_2}{2}x^2\right)$$

Also, by means of (17), the energy (9) reads

$$E = -3g_2 ,$$

while the potential (12) takes the form

$$V(x) = x^6 - 2g_2x^4 + (g_2^2 - 5)x^2 \tag{19}$$

Since the potential (19) is real and the wave function has only one node, it is the first-excited-state wave function [13]. We have thus obtained the first-excited-state wave function and energy of the real, quasi-exactly solvable sextic oscillator for $n = 1$, in line with [12].

Generally, the condition (18) is satisfied if g_1 and g_3 are imaginary and g_2 is real, i.e. it is met by PT-symmetric polynomials $g_4(x)$. Substituting the condition (18) and $n = 1$ into (12) yields

$$\begin{aligned} V(x) &= x^6 - 2g_3x^5 + (g_3^2 - 2g_2)x^4 + 4g_2g_3x^3 + (g_2^2 - 2g_2g_3^2 - 5)x^2 \\ &+ 2g_3(1 - g_2^2)x \end{aligned} \tag{20}$$

The potential (20) is a complex PT-symmetric sextic oscillator that converts to the real sextic oscillator (19) if g_3 vanishes. Using the condition (18), the polynomial $g_4(x)$ is written as

$$g_4(x) = -\frac{1}{4}x^4 + \frac{g_3}{3}x^3 + \frac{g_2}{2}x^2 - g_2g_3x,$$

and then, using also (16), the wave function (1) reads

$$\psi(x) = A(x - g_3)\exp\left(-\frac{1}{4}x^4 + \frac{g_3}{3}x^3 + \frac{g_2}{2}x^2 - g_2g_3x\right) \tag{21}$$

The wave function (21), which is odd under PT symmetry (since g_3 is imaginary), is energy eigenfunction of the oscillator (20), with energy given by (9), which, by means of (17) and (18), reads

$$E = -(g_2^2g_3^2 + 3g_2) \tag{22}$$

As noted, if g_3 vanishes, g_1 must also vanish for the condition (18) to be met, but the opposite does not necessarily hold, since if g_1 vanishes, then (18) is met if g_3 does not vanish, provided that g_2 vanishes, and then the oscillator (20) reads

$$V(x) = x^6 - 2g_3x^5 + g_3^2x^4 - 5x^2 + 2g_3x,$$

and it is again complex. The known energy eigenfunction of the previous PT-symmetric sextic oscillator is, by means of (21),

$$\psi(x) = A(x - g_3)\exp\left(-\frac{1}{4}x^4 + \frac{g_3}{3}x^3\right),$$

with zero energy, as seen from (22).

3.3 The case n=2

The polynomial $p_2(x)$ is monic and of degree 2, thus it has the form

$$p_2(x) = x^2 + p_1x + p_0,$$

and then (11) reads

$$2+2\left(-x^3+g_3x^2+g_2x+g_1\right)\left(2x+p_1\right)=-\left(4x^2+q_0\left(2\right)\right)\left(x^2+p_1x+p_0\right)$$

Equating the coefficients of the same-degree terms in x on both sides of the previous equation then yields

$$2g_3-p_1=-2p_1 \tag{23}$$

$$2\left(g_3p_1+2g_2\right)=-\left(4p_0+q_0\left(2\right)\right) \tag{24}$$

$$2\left(g_2p_1+2g_1\right)=-q_0\left(2\right)p_1 \tag{25}$$

$$2\left(g_1p_1+1\right)=-q_0\left(2\right)p_0 \tag{26}$$

Solving (23) for p_1 yields

$$p_1=-2g_3 \tag{27}$$

Substituting (27) into (24) and solving for p_0 then yields

$$p_0=g_3{}^2-g_2-\frac{q_0\left(2\right)}{4} \tag{28}$$

Also, substituting (27) into (25) yields

$$q_0\left(2\right)g_3=-2g_2g_3+2g_1 \tag{29}$$

To solve (29), we distinguish the cases $g_3=0$ and $g_3\neq 0$.

i. If $g_3=0$, from (29) we see that g_1 also vanishes, and then the polynomial $g_4\left(x\right)$ is real. Then, from (27) we derive that p_1 vanishes too, while (28) reads

$$p_0=-g_2-\frac{q_0\left(2\right)}{4} \tag{30}$$

and then (26) is written as

$$q_0{}^2\left(2\right)+4g_2q_0\left(2\right)-8=0,$$

and solving for $q_0\left(2\right)$ yields

$$q_{0\pm}\left(2\right)=-2g_2\pm 2\sqrt{g_2{}^2+2} \tag{31}$$

Substituting (31) into (9) and taking into account that g_1 vanishes, we obtain the energies

$$E_\pm=-3g_2\pm 2\sqrt{g_2{}^2+2} \tag{32}$$

Also, substituting (31) into (30) yields

$$p_0=-\frac{g_2\pm\sqrt{g_2{}^2+2}}{2}=-\frac{\left(g_2\pm\sqrt{g_2{}^2+2}\right)\left(g_2\mp\sqrt{g_2{}^2+2}\right)}{2\left(g_2\mp\sqrt{g_2{}^2+2}\right)}=\frac{1}{g_2\mp\sqrt{g_2{}^2+2}}$$

That is

$$p_0 = \frac{1}{g_2 \mp \sqrt{g_2^{\ 2} + 2}}$$

Then, since p_1 vanishes, we obtain the following two monic polynomials $p_{2\pm}(x)$

$$p_{2\pm}(x) = x^2 + \frac{1}{g_2 \mp \sqrt{g_2^{\ 2} + 2}} \tag{33}$$

Both polynomials (33) are of even parity. Also, since $g_2^{\ 2} + 2 > g_2^{\ 2}$, then $\sqrt{g_2^{\ 2} + 2} > |g_2| = \pm g_2$, and then we obtain $g_2 - \sqrt{g_2^{\ 2} + 2} < 0$ (from the inequality with the plus sign) and $g_2 + \sqrt{g_2^{\ 2} + 2} > 0$ (from the inequality with the minus sign). Thus, in (33), the polynomial $p_{2+}(x)$ has two real roots, while the polynomial $p_{2-}(x)$ is positive in $\mathbb{R}$, i.e. it has no real roots. Then, using (33) and that g_1 and g_3 vanish, the wave function (1) reads

$$\psi_{\pm}(x) = A_{\pm}\left(x^2 + \frac{1}{g_2 \mp \sqrt{g_2^{\ 2} + 2}}\right)\exp\left(-\frac{1}{4}x^4 + \frac{g_2}{2}x^2\right)$$

while the potential (12) takes the form of the real sextic oscillator

$$V(x) = x^6 - 2g_2x^4 + \left(g_2^{\ 2} - 7\right)x^2$$

Since the oscillator is real, its eigenfunctions are governed by the node theorem [13]. Then, the wave function $\psi_+(x)$, which has two nodes, describes the second-excited state of the previous oscillator, with energy E_+ given by (32), while the wave function $\psi_-(x)$, which is nodeless, describes the ground state of the same oscillator, with energy $E_- < E_+$ given also by (32). We have thus obtained the ground and second-excited-state wave functions and energies of the real, quasi-exactly solvable sextic oscillator for $n = 2$, in line with [12].

ii. If $g_3 \neq 0$, then solving (29) for $q_0(2)$ yields

$$q_0(2) = -2g_2 + 2\frac{g_1}{g_3} \tag{34}$$

Substituting (34) into (28) yields

$$p_0 = g_3^{\ 2} - \frac{g_2}{2} - \frac{g_1}{2g_3} \tag{35}$$

Finally, substituting (27), (34), and (35) into (26) yields the condition

$$2g_2g_3^{\ 4} + 2g_1g_3^{\ 3} - \left(2 + g_2^{\ 2}\right)g_3^{\ 2} + g_1^{\ 2} = 0 \tag{36}$$

This is the condition for the quotient polynomial to be real and the resulting PT-symmetric sextic oscillator to be non-real ($g_3 \neq 0$) in the case $n=2$. Since g_1 and g_3 are imaginary and g_2 is real, then setting $g_1 = i\tilde{g}_1$ and $g_3 = i\tilde{g}_3$, with $\tilde{g}_1, \tilde{g}_3$ real, the condition (36) reads

$$2g_2\tilde{g}_3^{\;4} + 2\tilde{g}_1\tilde{g}_3^{\;3} + \left(2 + g_2^{\;2}\right)\tilde{g}_3^{\;2} - \tilde{g}_1^{\;2} = 0,$$

which is a real quartic equation in $\tilde{g}_3$ – if g_2 does not vanish – and depending on the domains of $\tilde{g}_1$ and g_2, it can have up to four real roots, which are then expressed in terms of $\tilde{g}_1$ and g_2. Then, from (27) and (35), we determine the coefficients of the polynomials $p_2(x)$ and then from (1), we obtain the respective wave functions, while from (34) we calculate $q_0(2)$ and substituting into (9), we obtain the energies of the PT-symmetric sextic oscillators that are derived from (12).

We see that, in the case $n=2$, the set of PT-symmetric sextic oscillators resulting from real quotient polynomials is richer than in the case $n=1$.

3.4 The case n=3

The polynomial $p_3(x)$ is monic and of degree 3, thus it has the form

$$p_3(x) = x^3 + p_2x^2 + p_1x + p_0,$$

and then (11) reads

$$6x + 2p_2 + 2\left(-x^3 + g_3x^2 + g_2x + g_1\right)\left(3x^2 + 2p_2x + p_1\right) = -\left(6x^2 + q_0(3)\right)\left(x^3 + p_2x^2 + p_1x + p_0\right)$$

Equating the coefficients of the same-degree terms in x on both sides of the previous equation then yields

$$3g_3 - 2p_2 = -3p_2 \tag{37}$$

$$2\left(3g_2 + 2g_3p_2 - p_1\right) = -\left(6p_1 + q_0(3)\right) \tag{38}$$

$$2\left(3g_1 + g_3p_1 + 2g_2p_2\right) = -\left(6p_0 + q_0(3)\,p_2\right) \tag{39}$$

$$2\left(g_2p_1 + 2g_1p_2 + 3\right) = -q_0(3)\,p_1 \tag{40}$$

$$2\left(p_2 + g_1p_1\right) = -q_0(3)\,p_0 \tag{41}$$

Solving (37) for p_2 yields

$$p_2 = -3g_3 \tag{42}$$

Substituting (42) into (38) and solving for p_1 yields

$$p_1 = -\frac{3g_2}{2} + 3g_3^{\;2} - \frac{q_0(3)}{4} \tag{43}$$

Substituting (42) and (43) into (39) and solving for p_0 yields

$$p_0 = -g_1 + \frac{15 g_2 g_3}{6} - g_3{}^3 + \frac{7 g_3 q_0(3)}{12} \tag{44}$$

Besides, substituting (42) and (43) into (40) yields

$$q_0{}^2(3) + 4\left(2g_2 - 3g_3{}^2\right) q_0(3) + 12 g_2{}^2 - 24 g_2 g_3{}^2 + 48 g_1 g_3 - 24 = 0 \tag{45}$$

Also, substituting (42), (43), and (44) into (41) yields

$$7 g_3 q_0{}^2(3) + \left(30 g_2 g_3 - 18 g_1 - 12 g_3{}^3\right) q_0(3) - 72 g_3 - 36 g_1 g_2 + 72 g_1 g_3{}^2 = 0 \tag{46}$$

The equations (45) and (46) must be satisfied simultaneously. As in the case $n = 2$, we distinguish the cases $g_3 = 0$ and $g_3 \neq 0$.

i. If $g_3 = 0$, (45) and (46) read, respectively,

$$q_0{}^2(3) + 8 g_2 q_0(3) + 12 g_2{}^2 - 24 = 0 \tag{47}$$

$$g_1 q_0(3) + 2 g_1 g_2 = 0 \tag{48}$$

Then, if g_1 does not vanish, (48) gives $q_0(3) = -2g_2$ and substituting into (47) yields $-24 = 0$, which is impossible. Thus, if g_3 vanishes, then g_1 vanishes too. Then, (48) holds identically and we are left only with (47), which, if solved for $q_0(3)$, yields

$$q_{0\pm}(3) = -4 g_2 \pm 2\sqrt{g_2{}^2 + 6} \tag{49}$$

Then, substituting into (9), we obtain the energies

$$E_{\pm} = -5 g_2 \pm 2\sqrt{g_2{}^2 + 6} \tag{50}$$

Also, since g_1 and g_3 vanish, from (42) and (44) we respectively see that p_2 and p_0 also vanish, while, from (43), p_1 reads

$$p_1 = -\frac{3 g_2}{2} - \frac{q_{0\pm}(3)}{4}$$

Then, we obtain the following two monic polynomials $p_{3\pm}(x)$

$$p_{3\pm}(x) = x\left(x^2 - \left(\frac{3 g_2}{2} + \frac{q_{0\pm}(3)}{4}\right)\right),$$

which are real and of odd parity, and so are the respective wave functions (1), since, in this case, the polynomial $g_4(x)$ is real and of even parity. Also, by means of (49), we derive that

$$-\left(\frac{3 g_2}{2} + \frac{q_{0\pm}(3)}{4}\right) = -\frac{g_2 \pm \sqrt{g_2{}^2 + 6}}{2} = -\frac{\left(g_2 \pm \sqrt{g_2{}^2 + 6}\right)\left(g_2 \mp \sqrt{g_2{}^2 + 6}\right)}{2\left(g_2 \mp \sqrt{g_2{}^2 + 6}\right)},$$

$$= \frac{3}{g_2 \mp \sqrt{g_2{}^2 + 6}}$$

and thus

$$p_{3\pm}(x) = x\left(x^2 + \frac{3}{g_2 \mp \sqrt{g_2^{\,2}+6}}\right)$$

Since $\sqrt{g_2^{\,2}+6} > |g_2| = \pm g_2$, then $g_2 - \sqrt{g_2^{\,2}+6} < 0$ (from the inequality with the plus sign) and $g_2 + \sqrt{g_2^{\,2}+6} > 0$ (from the inequality with the minus sign). Thus, the polynomial $p_{3+}(x)$, corresponding to the energy $E_+ > E_-$ has three real roots, while the polynomial $p_{3-}(x)$, corresponding to the energy E_-, has one real root. Then, from (1), we obtain the wave functions

$$\psi_\pm(x) = A_\pm p_{3\pm}(x)\exp\left(-\frac{1}{4}x^4 + \frac{g_2}{2}x^2\right),$$

where $\psi_+(x)$ corresponds to the energy E_+ and it has three nodes, while $\psi_-(x)$ corresponds to the energy E_- and it has one node. The potential is obtained from (12) if we set $n=3$ and take into account that both g_1 and g_3 vanish, thus

$$V(x) = x^6 - 2g_2x^4 + \left(g_2^{\,2} - 9\right)x^2$$

This is a real sextic oscillator and by the node theorem [13], the above wave functions describe, respectively, its third and first-excited states, with energies being given by (50), respectively. We have thus obtained the first and third-excited-state wave functions and energies of the real, quasi-exactly solvable sextic oscillator for $n=3$, in line with [12].

ii. If $g_3 \neq 0$, then (46) is written as

$$q_0^{\,2}(3) + \left(\frac{30g_2}{7} - \frac{18g_1}{7g_3} - \frac{12g_3^{\,2}}{7}\right)q_0(3) - \frac{72}{7} - \frac{36g_1g_2}{7g_3} + \frac{72g_1g_3}{7} = 0 \qquad (51)$$

Subtracting (51) from (45) then yields

$$\left(13g_2g_3 - 36g_3^{\,3} + 9g_1\right)q_0(3) + 42g_2^{\,2}g_3 - 84g_2g_3^{\,3} + 132g_1g_3^{\,2} - 48g_3 + 18g_1g_2 = 0 \qquad (52)$$

If $13g_2g_3 - 36g_3^{\,3} + 9g_1 \neq 0$, solving (52) for $q_0(3)$ yields

$$q_0(3) = \frac{-42g_2^{\,2}g_3 + 84g_2g_3^{\,3} - 132g_1g_3^{\,2} + 48g_3 - 18g_1g_2}{13g_2g_3 - 36g_3^{\,3} + 9g_1} \qquad (53)$$

Then, substituting (53) into (45), we obtain the condition

$$\begin{aligned}&\left(-21g_2^{\,2}g_3 + 42g_2g_3^{\,3} - 66g_1g_3^{\,2} + 24g_3 - 9g_1g_2\right)^2\\ &+\left(2g_2 - 3g_3^{\,2}\right)\left(13g_2g_3 - 36g_3^{\,3} + 9g_1\right)\left(-42g_2^{\,2}g_3 + 84g_2g_3^{\,3} - 132g_1g_3^{\,2} + 48g_3 - 18g_1g_2\right) \quad (54)\\ &+\left(13g_2g_3 - 36g_3^{\,3} + 9g_1\right)^2\left(3g_2^{\,2} - 6g_2g_3^{\,2} + 12g_1g_3 - 6\right) = 0\end{aligned}$$

Since g_1 and g_3 are imaginary and g_2 is real, we set $g_1 = i\tilde{g}_1$ and $g_3 = i\tilde{g}_3$, with $\tilde{g}_1, \tilde{g}_3$ real, and the condition (54) takes the form of the following real algebraic equation of degree eight in $\tilde{g}_3$

$$\left(-21g_2{}^2\tilde{g}_3-42g_2\tilde{g}_3{}^3+66\tilde{g}_1\tilde{g}_3{}^2+24\tilde{g}_3-9\tilde{g}_1g_2\right)^2$$
$$+\left(2g_2+3\tilde{g}_3{}^2\right)\left(13g_2\tilde{g}_3+36\tilde{g}_3{}^3+9\tilde{g}_1\right)\left(-42g_2{}^2\tilde{g}_3-84g_2\tilde{g}_3{}^3+132\tilde{g}_1\tilde{g}_3{}^2+48\tilde{g}_3-18\tilde{g}_1g_2\right) \quad (55)$$
$$+\left(13g_2\tilde{g}_3+36\tilde{g}_3{}^3+9\tilde{g}_1\right)^2\left(3g_2{}^2+6g_2\tilde{g}_3{}^2-12\tilde{g}_1\tilde{g}_3-6\right)=0$$

It is easily seen that the leading coefficient of the eighth-degree polynomial in $\tilde{g}_3$ on the left-hand side of (55) is $-1296g_2$. Then, depending on the domains of $\tilde{g}_1$ and g_2, the equation (55) can have up to eight real roots, which are then expressed in terms of $\tilde{g}_1$ and g_2. Then, from (42), (43), and (44) we derive the coefficients of the polynomials $p_3(x)$ and then from (1), we obtain the respective wave functions, while from (53) we calculate $q_0(3)$ and substituting into (9), we obtain the energies of the PT-symmetric sextic oscillators that are derived from (12).

We see that, in the case $n=3$, the set of PT-symmetric sextic oscillators resulting from real quotient polynomials is richer than in the case $n=2$.

4. Conclusions

We have presented a method of constructing PT-symmetric sextic oscillators using quotient polynomials and demonstrated the binding relation between the reality of the energy spectrum of the oscillators and the PT invariance of the respective quotient polynomials. We have then used real quotient polynomials to derive PT-symmetric oscillators in the cases where the non-negative integer parameter n ranges from 0 up to 3, and showed that the set of resulting oscillators, which contains the real, quasi-exactly solvable sextic oscillator as a special case, becomes richer as n increases, a property rendering the system quasi-exactly solvable. As these oscillators are endowed with a special characteristic, it is natural to distinguish them from those resulting from non-real quotient polynomials and study them separately.

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