Quasi-exact solution of sextic anharmonic oscillator using a quotient polynomial

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Abstract

Among the one-dimensional, real and analytic polynomial potentials, the sextic anharmonic oscillator is the only one that can be quasi-exactly solved, if it is properly parametrized. In this work, we present a new method to quasi-exactly solve the sextic anharmonic oscillator and apply it to derive specific solutions. Our approach is based on the introduction of a quotient polynomial and can also be used to study the solvability of symmetrized (non-analytic) or complex PT-symmetric polynomial potentials, where it opens up new options.

***Keywords*:** quasi-exactly solvable potentials, sextic anharmonic oscillator, quotient polynomial, energy-reflection symmetry

1. Introduction

The search for exact solutions to the Schrödinger equation has resulted in the discovery of a new class of potentials, for which only a finite part of the energy spectrum, along with the respective eigen functions, can be found in closed-form [1-4]. These potentials are called quasi-exactly solvable (QES) and occupy an intermediate place, between the few which are exactly solvable, such as the harmonic oscillator and the Coulomb potential, for which the entire energy spectrum and all eigen functions are known, and the many which are non-solvable, for which none eigen value and none eigen function can be exactly determined.

The sextic anharmonic oscillator is the only one-dimensional, real and analytic polynomial potential that can be quasi-exactly solved if its parameters are properly chosen [2]. Depending on the coupling constant, the system can be a single, double, or triple-well potential. Single and double-well oscillators are among the simplest interacting quantum systems with numerous applications in diverse areas of physics and chemistry, ranging from atomic and molecular physics to particle physics, quantum field theory, and cosmology. For instance, as single-well potentials, sextic anharmonic oscillators can be used to verify results derived from approximation methods, give a better insight into the φ^6 quantum field theory, and model nonlinear fields, while as double-well potentials, they are useful to describe the motion of a particle in the presence of two centers of force, Bose-Einstein condensates, quantum heterostructures, chemical processes in molecular systems, quantum tunneling effects, and so on.

Herein, we adopt an approach that uses elements of both the Bethe ansatz method [5, 6] and the recursion relation method [7] to study the sextic anharmonic oscillator by means of a quotient polynomial.

The rest of the paper is organized as follows: in the next section, making an ansatz for the wave function, we introduce the quotient polynomial, transform the Schrödinger equation into a differential equation containing the quotient polynomial, and derive the sextic anharmonic oscillator to be quasi-exactly solved. In sections 3 and 4, we quasi-exactly solve the system in the cases of even and odd parity of the wave function, and give examples of specific solutions. In section 5, we show that the system exhibits energy-reflection symmetry if the coupling constant vanishes and, in section 6, we summarize and outline future developments.

2. Introducing the quotient polynomial

Choosing a length scale l and doing the transformations $x \to lx$, $E \to \hbar^2 E / 2ml^2$, and $V(x) \to \hbar^2 V(x) / 2ml^2$, the position x, the energy E, and the potential $V(x)$ become dimensionless, and then the energy eigenvalue equation – i.e. the time-independent Schrödinger equation – for our system reads

$$\psi''(x) + \left(E - V(x)\right)\psi(x) = 0,$$

where $\psi(x)$ is an energy eigenfunction of the system.

Establishing our ansatz scheme, we seek definite-parity eigenfunctions of the form

$$\psi(x) = A_n p_n(x) \exp\left(g_4(x)\right) \tag{1}$$

where A_n the normalization constant, $p_n(x)$ an n-degree polynomial of definite parity, and $g_4(x)$ a fourth-degree polynomial of even parity with negative leading coefficient, so that (1) is square-integrable, which is necessary for a confining potential. Since x is dimensionless, the dimensions of $p_n(x)$ is carried by its coefficients. Thus, incorporating the leading coefficient of $p_n(x)$ into the normalization constant A_n, we make $p_n(x)$ both monic and dimensionless. As exponent, the polynomial $g_4(x)$ must be dimensionless too, and since x is dimensionless, the coefficients of $g_4(x)$ are also dimensionless. The constant term of $g_4(x)$ is a multiplicative constant that can also be incorporated into the normalization constant A_n. Finally, choosing the length scale l appropriately, we can set the leading coefficient of $g_4(x)$ to a desirable negative value. Then, without loss of generality, we write $g_4(x)$ as

$$g_4(x) = -\frac{1}{4}x^4 + \frac{g_2}{2}x^2 \tag{2}$$

where g_2 is a real parameter.

The second derivative of (1) with respect to the dimensionless x is

$$\psi''(x) = A_n\left(p_n''(x) + 2g_4'(x)\,p_n'(x) + \left(g_4'^2(x) + g_4''(x)\right)p_n(x)\right)\exp\left(g_4(x)\right)$$

Plugging the previous expression along with (1) into the energy eigenvalue equation, then dividing by $A_n \exp\left(g_4(x)\right) \neq 0$ and solving for the potential, we end up to

$$V(x) = \frac{p_n''(x) + 2g_4'(x)\,p_n'(x)}{p_n(x)} + g_4'^2(x) + g_4''(x) + E$$

Since the potential $V(x)$ and $g_4'^2(x) + g_4''(x) + E$ are both polynomials, the expression $\left(p_n''(x) + 2g_4'(x)\,p_n'(x)\right)\Big/ p_n(x)$ is a polynomial too, and it is quadratic, as

$$\deg\left(\left(p_n'' + 2g_4'p_n'\right)\Big/ p_n\right) = \deg\left(p_n'' + 2g_4'p_n'\right) - \deg\left(p_n\right) = \deg\left(g_4'p_n'\right) - \deg\left(p_n\right) =$$
$$= \deg\left(g_4'\right) + \deg\left(p_n'\right) - \deg\left(p_n\right) = 3 + n - 1 - n = 2$$

Also, $g_4'(x)$ is of odd parity and $p_n'(x)$ has different parity from $p_n(x)$, thus $g_4'(x)\,p_n'(x)$ has the same parity as $p_n(x)$, and $p_n''(x)$ has also the same parity as $p_n(x)$, and then

$p_n''(x)+2g_4'(x)p_n'(x)$ has also the same parity as $p_n(x)$, and thus the polynomial $\left(p_n''(x)+2g_4'(x)p_n'(x)\right)/p_n(x)$ is of even parity. Therefore, we can write

$$p_n''(x)+2g_4'(x)p_n'(x)=-q_2(x;n)p_n(x) \tag{3}$$

where

$$q_2(x;n)=q_2(n)x^2+q_0(n) \tag{4}$$

We'll refer to $q_2(x;n)$ as the quotient polynomial. The minus sign on the right-hand side of (3) is put in for convenience.

Using (2) and (4), (3) takes the form

$$p_n''(x)+2\left(-x^3+g_2x\right)p_n'(x)=-\left(q_2(n)x^2+q_0(n)\right)p_n(x) \tag{5}$$

Besides, equating the coefficients of the highest-order terms in x on both sides of (5) yields

$$q_2(n)=2n \tag{6}$$

Then, the quotient polynomial (4) and the differential equation (5) become, respectively,

$$q_2(x;n)=2nx^2+q_0(n) \tag{7}$$

$$p_n''(x)+2\left(-x^3+g_2x\right)p_n'(x)=-\left(2nx^2+q_0(n)\right)p_n(x) \tag{8}$$

In terms of the quotient polynomial, the expression of the potential takes the form

$$V(x)=-q_2(x;n)+g_4'^2(x)+g_4''(x)+E \tag{9}$$

The potential (9) is expressed up to an additive constant and to determine it uniquely, we choose its value at zero to be zero. Then, using the expressions of the quotient polynomial and of $g_4(x)$, we obtain that the energy is given by

$$E=q_0(n)-g_2 \tag{10}$$

Substituting (10) into (9) and taking into account the expressions of the quotient polynomial and $g_4(x)$, we end up to the potential

$$V(x)=x^6-2g_2x^4+\left(g_2^2-(3+2n)\right)x^2 \tag{11}$$

with $n=0,1,...$ For every value of n, the potential (11) describes a symmetric sextic anharmonic oscillator that, as we'll show, is quasi-exactly solvable.

Before finishing this section, it is worth mentioning that the vanishing of the linear term $q_1(n)$ in the quotient polynomial results in the potential (11) being dependent only on n and not on the energy. Otherwise, $q_1(n)$ would appear in (11) and its dependence on the energy would result in the potential being also dependent on the energy, which is generally unwanted, because if the potential changes with the energy, then only one eigenstate of it could be found.

3. The even-parity case

We'll examine first the even-parity case for the polynomial $p_n(x)$.

Since the polynomial $p_n(x)$ is of n degree, n must be even. Also, since the exponential factor $\exp(g_4(x))$ is of even parity, the ansatz eigenfunction (1) is also of even parity in this case. Setting, for convenience, $n \to 2n$, with $n = 0,1,\ldots$, the oscillator (11) takes the form

$$V(x) = x^6 - 2g_2x^4 + \left(g_2{}^2 - (3+4n)\right)x^2 \tag{12}$$

while the differential equation (8) is written as

$$p_{2n}''(x) + 2\left(-x^3 + g_2x\right)p_{2n}'(x) = -\left(4nx^2 + q_0(2n)\right)p_{2n}(x) \tag{13}$$

where

$$p_{2n}(x) = x^{2n} + p_{2n-2}x^{2n-2} + \ldots + p_0$$

The product $2\left(-x^3 + g_2x\right)p_{2n}'(x)$ is of even parity, as are all other polynomials in (13), and thus we have only even-degree terms in x. Then, equating the coefficients of the $2k$-degree terms in x on both sides of (13) yields the following three-term recursion relation

$$(2k+2)(2k+1)p_{2k+2} = -\left(4kg_2 + q_0(2n)\right)p_{2k} - 4(n+1-k)p_{2k-2} \tag{14}$$

For $k=0$, we drop the coefficient p_{2k-2}, whose index is negative, and (14) gives

$$p_2 = -\frac{q_0(2n)}{2}p_0 \tag{15}$$

For $k = n+1$, dropping the coefficients p_{2k} and p_{2k+2}, whose indices exceed the degree of $p_{2n}(x)$, we see that (14) holds identically, which is expected, since for $k = n+1$, from (14) we calculate the leading coefficient of the quotient polynomial, i.e. the relation (6) for $n \to 2n$. Then, we'll use (14) for $k = 0,1,\ldots,n$.

Using the $n+1$ different values of k, we obtain a system of $n+1$ equations with the $n+1$ unknowns $p_0, p_{2*1}, ..., p_{2*(n-1)}$, and $q_0(2n)$.

The relation (15) is written as

$$p_{2*1} = f_1(q_0(2n))\, p_0$$

with $f_1(q_0(2n)) = -q_0(2n)/2$, a polynomial of degree 1 in $q_0(2n)$. Using the previous expression of p_2, we obtain from (14), for $k=1$,

$$4*3p_4 = -\left(\left(4g_2 + q_0(2n)\right) f_1(q_0(2n)) + 4n\right) p_0$$

Now, setting

$$-\left(\left(4g_2 + q_0(2n)\right) f_1(q_0(2n)) + 4n\right) = 4*3 f_2(q_0(2n)),$$

the previous expression of p_4 is written as

$$p_{2*2} = f_2(q_0(2n))\, p_0,$$

with $f_2(q_0(2n))$ a polynomial of degree 2 in $q_0(2n)$.

We assume that

$p_{2*(k-1)} = f_{k-1}(q_0(2n))\, p_0$ and $p_{2*k} = f_k(q_0(2n))\, p_0$,

with $f_{k-1}(q_0(2n))$ and $f_k(q_0(2n))$ polynomials of degree $k-1$ and k in $q_0(2n)$, respectively. Then, substituting into (14), we obtain

$$(2k+2)(2k+1)\, p_{2k+2} = -\left(\left(4kg_2 + q_0(2n)\right) f_k(q_0(2n)) + 4(n+1-k) f_{k-1}(q_0(2n))\right) p_0$$

Setting

$$-\left(\left(4kg_2 + q_0(2n)\right) f_k(q_0(2n)) + 4(n+1-k) f_{k-1}(q_0(2n))\right) = (2k+2)(2k+1) f_{k+1}(q_0(2n)),$$

the previous expression of p_{2k+2} is written as

$$p_{2(k+1)} = f_{k+1}(q_0(2n))\, p_0,$$

with $f_{k+1}(q_0(2n))$ a polynomial of degree $k+1$ in $q_0(2n)$.

Then, by induction,

$$p_{2k} = f_k(q_0(2n))\, p_0 \qquad (16)$$

for $k=1,2,...n$, $n=0,1,...$, with $f_k\left(q_0\left(2n\right)\right)$ a polynomial of degree k in $q_0\left(2n\right)$, and $f_1\left(q_0\left(2n\right)\right)=-q_0\left(2n\right)/2$. From (16), we see that if p_0 vanishes, all coefficients of $p_{2n}\left(x\right)$ vanish, which is impossible, since $p_{2n}\left(x\right)$ is monic. Thus $p_0\neq 0$, and then (16) holds for $k=0$ if $f_0\left(q_0\left(2n\right)\right)=1$. Thus, (16) holds for $k=0,1,...n$, with $f_0\left(q_0\left(2n\right)\right)=1$ and $f_1\left(q_0\left(2n\right)\right)=-q_0\left(2n\right)/2$.

Using (16), the recursion relation (14) is written as

$$\left(2k+2\right)\left(2k+1\right)f_{k+1}\left(q_0\left(2n\right)\right)=-\left(4kg_2+q_0\left(2n\right)\right)f_k\left(q_0\left(2n\right)\right)-4\left(n+1-k\right)f_{k-1}\left(q_0\left(2n\right)\right) \quad (17)$$

where we divided both members by $p_0\neq 0$. The recursion relation (17) holds for $k=0,1,...,n$, $n=0,1,...$, with $f_0\left(q_0\left(2n\right)\right)=1$ and $f_1\left(q_0\left(2n\right)\right)=-q_0\left(2n\right)/2$. For $k=n$, we drop $f_{k+1}\left(q_0\left(2n\right)\right)$, as it is related to the coefficient $p_{2(k+1)}$ whose index exceeds the degree of $p_{2n}\left(x\right)$, and we obtain

$$q_0\left(2n\right)f_n\left(q_0\left(2n\right)\right)+4ng_2f_n\left(q_0\left(2n\right)\right)+4f_{n-1}\left(q_0\left(2n\right)\right)=0 \quad (18)$$

The equation (18), which is a $\left(n+1\right)$-degree polynomial equation, gives the values of $q_0\left(2n\right)$ and then, from (10), the energies of the known eigenstates of the sextic anharmonic oscillator (12). The equation (18) is equivalently written as

$$f_{n+1}\left(q_0\left(2n\right)\right)=0 \quad (19)$$

We'll refer to (18) or (19) as the energy equation.

Next, we'll show that the polynomial $f_k\left(q_0\left(2n\right)\right)$ has k (real) zeros.

Since $f_0\left(q_0\left(2n\right)\right)=1$ (no zeros) and $f_1\left(q_0\left(2n\right)\right)=-q_0\left(2n\right)/2$ (one zero, at zero), it is enough to show that $f_k\left(q_0\left(2n\right)\right)$ has k zeros for $k\geq 2$.

From (17), we see that the leading coefficients of $f_k\left(q_0\left(2n\right)\right)$ and $f_{k+1}\left(q_0\left(2n\right)\right)$ have opposite signs. Then, since the leading coefficient of $f_0\left(q_0\left(2n\right)\right)$ is positive, we derive that $f_k\left(q_0\left(2n\right)\right)$ has positive/negative leading coefficient if (and only if) k is even/odd. As a consequence, the leading coefficients of $f_{k-1}\left(q_0\left(2n\right)\right)$ and $f_{k+1}\left(q_0\left(2n\right)\right)$ have the same sign. Also, since $f_k\left(q_0\left(2n\right)\right)$ is a polynomial of degree k in $q_0\left(2n\right)$, if k is even, $f_k\left(\pm\infty\right)=\infty$, and thus $\operatorname{sgn}\left(f_k\left(\pm\infty\right)\right)=1$, while if k is odd, $f_k\left(\pm\infty\right)=\mp\infty$, and thus $\operatorname{sgn}\left(f_k\left(\pm\infty\right)\right)=\mp 1$. We observe that in both cases, $\operatorname{sgn}\left(f_k\left(-\infty\right)\right)=1$.

For $k=2$, we have $\operatorname{sgn}\left(f_2\left(\pm\infty\right)\right)=1$. Also, (17) gives for $k=1$, at $q_0\left(2n\right)=0$, $12f_2\left(0\right)=-4n<0$, since $f_1\left(0\right)=0$, $f_0\left(0\right)=1$, and $n>0$, because if $n=0$, $f_2\left(q_0\left(2n\right)\right)$ vanishes. Thus $f_2\left(0\right)<0$, and then $f_2\left(q_0\left(2n\right)\right)$ changes sign in the interval $\left(-\infty,0\right)$ and in the interval $\left(0,\infty\right)$. Then, since $f_2\left(q_0\left(2n\right)\right)$ is a second degree polynomial, it has exactly one zero in each of the two previous intervals. Thus, $f_2\left(q_0\left(2n\right)\right)$ has exactly one zero in each of the two intervals that the zero of $f_1\left(q_0\left(2n\right)\right)$ splits the real line.

We'll now show, by induction, that, for $k\geq 2$, $f_k\left(q_0\left(2n\right)\right)$ has exactly one zero in each of the k intervals that the $k-1$ zeros of $f_{k-1}\left(q_0\left(2n\right)\right)$ split the real line.

Proof

For $k=2$, we showed that the previous statement holds.

We assume that $f_{k-1}\left(q_0\left(2n\right)\right)$ has $k-1$ simple zeros, with $k-1\geq 1$, which thus split the real line into k intervals, and in each of these intervals, $f_k\left(q_0\left(2n\right)\right)$ has a zero. Since $f_k\left(q_0\left(2n\right)\right)$ is a polynomial of k degree, it has exactly one zero in each of the previous k intervals. Denoting the zeros of $f_{k-1}\left(q_0\left(2n\right)\right)$ by $q_0^{(k-1,i)}\left(2n\right)$, with $i=1,2,...,k-1$, without loss of generality, we arrange them as

$$-\infty<q_0^{(k-1,1)}\left(2n\right)<q_0^{(k-1,2)}\left(2n\right)<...<q_0^{(k-1,k-1)}\left(2n\right)<\infty$$

Then, in each of the k intervals

$$\left(-\infty,q_0^{(k-1,1)}\left(2n\right)\right),\left(q_0^{(k-1,1)}\left(2n\right),q_0^{(k-1,2)}\left(2n\right)\right),...,\left(q_0^{(k-1,k-2)}\left(2n\right),q_0^{(k-1,k-1)}\left(2n\right)\right),\left(q_0^{(k-1,k-1)}\left(2n\right),\infty\right),$$

the polynomial $f_{k-1}\left(q_0\left(2n\right)\right)$ has constant sign. Since the zeros are simple, the derivative $df_{k-1}\left(q_0\left(2n\right)\right)/dq_0\left(2n\right)$ (with respect to q_0) does not vanish at the zeros of $f_{k-1}\left(q_0\left(2n\right)\right)$, and then the zeros of $f_{k-1}\left(q_0\left(2n\right)\right)$ are not extreme points of $f_{k-1}\left(q_0\left(2n\right)\right)$. Thus, in every two consecutive intervals of the above k intervals, the polynomial $f_{k-1}\left(q_0\left(2n\right)\right)$ has opposite signs, and thus if q_0' and q_0'' belong, respectively, to any two consecutive intervals of the above k intervals, then $f_{k-1}\left(q_0'\right)f_{k-1}\left(q_0''\right)<0$.

Arranging, without loss of generality, the k zeros of $f_k\left(q_0\left(2n\right)\right)$ as

$$-\infty < q_0^{(k,1)}(2n) < q_0^{(k,2)}(2n) < \ldots < q_0^{(k,k)}(2n) < \infty ,$$

we have

$$-\infty < q_0^{(k,1)}(2n) < q_0^{(k-1,1)}(2n) < q_0^{(k,2)}(2n) < q_0^{(k-1,2)}(2n) < \ldots$$
$$\ldots < q_0^{(k,k-1)}(2n) < q_0^{(k-1,k-1)}(2n) < q_0^{(k,k)}(2n) < \infty$$

Then

$$f_{k-1}\left(q_0^{(k,i)}(2n)\right) f_{k-1}\left(q_0^{(k,i+1)}(2n)\right) < 0 ,$$

for every $i = 1, 2, \ldots k-1$.

Besides, for $n \geq k+1$, $f_{k+1}\left(q_0(2n)\right)$ is given by the recursion relation (17), which at $q_0^{(k,i)}(2n)$ and at $q_0^{(k,i+1)}(2n)$ is respectively written as

$$(2k+2)(2k+1) f_{k+1}\left(q_0^{(k,i)}(2n)\right) = -4(n+1-k) f_{k-1}\left(q_0^{(k,i)}(2n)\right)$$

$$(2k+2)(2k+1) f_{k+1}\left(q_0^{(k,i+1)}(2n)\right) = -4(n+1-k) f_{k-1}\left(q_0^{(k,i+1)}(2n)\right),$$

since $f_k\left(q_0^{(k,i)}(2n)\right) = f_k\left(q_0^{(k,i+1)}(2n)\right) = 0$. Multiplying side-by-side the two equations, we obtain

$$\left((2k+2)(2k+1)\right)^2 f_{k+1}\left(q_0^{(k,i)}(2n)\right) f_{k+1}\left(q_0^{(k,i+1)}(2n)\right) =$$
$$= 16(n+1-k)^2 f_{k-1}\left(q_0^{(k,i)}(2n)\right) f_{k-1}\left(q_0^{(k,i+1)}(2n)\right)$$

For $n \geq k+1$, $n+1-k \neq 0$, and since $f_{k-1}\left(q_0^{(k,i)}(2n)\right) f_{k-1}\left(q_0^{(k,i+1)}(2n)\right) < 0$, from the previous equation we derive that

$$f_{k+1}\left(q_0^{(k,i)}(2n)\right) f_{k+1}\left(q_0^{(k,i+1)}(2n)\right) < 0 ,$$

for every $i = 1, 2, \ldots., k-1$. Thus, $f_{k+1}\left(q_0(2n)\right)$ has at least one zero in each of the $k-1$ intervals

$$\left(q_0^{(k,1)}(2n), q_0^{(k,2)}(2n)\right), \left(q_0^{(k,2)}(2n), q_0^{(k,3)}(2n)\right), \ldots., \left(q_0^{(k,k-1)}(2n), q_0^{(k,k)}(2n)\right)$$

Besides, at $q_0^{(k,1)}(2n)$, the recursion relation (17) is written as

$$(2k+2)(2k+1) f_{k+1}\left(q_0^{(k,1)}(2n)\right) = -4(n+1-k) f_{k-1}\left(q_0^{(k,1)}(2n)\right)$$

For $n \geq k+1$, $n+1-k > 0$, and from the previous equation we derive that

$$\operatorname{sgn}\left(f_{k+1}\left(q_0^{(k,1)}(2n)\right)\right) = -\operatorname{sgn}\left(f_{k-1}\left(q_0^{(k,1)}(2n)\right)\right) \tag{20}$$

But $q_0^{(k,1)}(2n)$ belongs to the interval $\left(-\infty, q_0^{(k-1,1)}(2n)\right)$ in which $f_{k-1}(q_0(2n))$ has constant sign, and thus

$$\mathrm{sgn}\left(f_{k-1}\left(q_0^{(k,1)}(2n)\right)\right) = \mathrm{sgn}\left(f_{k-1}(-\infty)\right) = 1,$$

because, as shown above, $\mathrm{sgn}\left(f_k(-\infty)\right) = 1$ for all k. Then (20) gives

$$\mathrm{sgn}\left(f_{k+1}\left(q_0^{(k,1)}(2n)\right)\right) = -1,$$

and thus

$$\mathrm{sgn}\left(f_{k+1}(-\infty)\right)\mathrm{sgn}\left(f_{k+1}\left(q_0^{(k,1)}(2n)\right)\right) = -1$$

Then, in the interval $\left(-\infty, q_0^{(k,1)}(2n)\right)$, the polynomial $f_{k+1}(q_0(2n))$ changes sign, and particularly, it becomes from positive, negative, and thus $f_{k+1}(q_0(2n))$ has at least one zero in this interval.

Similarly, at $q_0^{(k,k)}(2n)$, the recursion relation (17) is written as

$$(2k+2)(2k+1) f_{k+1}\left(q_0^{(k,k)}(2n)\right) = -4(n+1-k) f_{k-1}\left(q_0^{(k,k)}(2n)\right),$$

and thus

$$\mathrm{sgn}\left(f_{k+1}\left(q_0^{(k,k)}(2n)\right)\right) = -\mathrm{sgn}\left(f_{k-1}\left(q_0^{(k,k)}(2n)\right)\right) \tag{21}$$

But $q_0^{(k,k)}(2n)$ belongs to the interval $\left(q_0^{(k-1,k-1)}(2n), \infty\right)$ in which $f_{k-1}(q_0(2n))$ has constant sign, and thus

$$\mathrm{sgn}\left(f_{k-1}\left(q_0^{(k,k)}(2n)\right)\right) = \mathrm{sgn}\left(f_{k-1}(\infty)\right)$$

The numbers $k-1$ and $k+1$ are of the same parity, and thus

$$\mathrm{sgn}\left(f_{k-1}(\infty)\right) = \mathrm{sgn}\left(f_{k+1}(\infty)\right),$$

because, as shown above, the leading coefficients of $f_{k-1}(q_0(2n))$ and $f_{k+1}(q_0(2n))$ have the same sign. Comparing the last two equations yields

$$\mathrm{sgn}\left(f_{k-1}\left(q_0^{(k,k)}(2n)\right)\right) = \mathrm{sgn}\left(f_{k+1}(\infty)\right),$$

and comparing (21) with the last equation yields

$$\mathrm{sgn}\left(f_{k+1}\left(q_0^{(k,k)}(2n)\right)\right) = -\mathrm{sgn}\left(f_{k+1}(\infty)\right),$$

and then

$$\operatorname{sgn}\left(f_{k+1}\left(q_0^{(k,k)}(2n)\right)\right)\operatorname{sgn}\left(f_{k+1}(\infty)\right)=-\operatorname{sgn}^2\left(f_{k+1}(\infty)\right)=-1$$

Then, in the interval $\left(q_0^{(k,k)}(2n),\infty\right)$, the polynomial $f_{k+1}\left(q_0(2n)\right)$ changes sign, and thus it has at least one zero in this interval.

We have thus showed that the polynomial $f_{k+1}\left(q_0(2n)\right)$ has at least one zero in each of the following $k+1$ intervals

$$\left(-\infty,q_0^{(k,1)}(2n)\right),\left(q_0^{(k,1)}(2n),q_0^{(k,2)}(2n)\right),....,\left(q_0^{(k,k-1)}(2n),q_0^{(k,k)}(2n)\right),\left(q_0^{(k,k)}(2n),\infty\right),$$

i.e. it has at least $k+1$ zeros. But, since it is a $(k+1)$-degree polynomial, it can have up to $k+1$ zeros. Thus, the polynomial $f_{k+1}\left(q_0(2n)\right)$ has exactly one zero in each of the above intervals and, in total, it has $k+1$ zeros, and this completes the proof.

For every $n=0,1,...$, solving (19), we obtain $n+1$ different values of $q_0(2n)$, and then from (10) we obtain $n+1$ different energies of the sextic anharmonic oscillator (12). Since the potential is one-dimensional, its bound states are non-degenerate[1] [8], and thus the $n+1$ energies correspond to $n+1$ eigenstates. To locate the eigenstates, we use the node theorem [9], which, in this case, where the potential is regular, implies that the ground-state wave function has no zeros, while the n th-excited-state wave function has n zeros.

Besides, as we showed, the constant term p_0 of the polynomial $p_{2n}(x)$ does not vanish, and thus $p_{2n}(x)$ does not have a zero at zero. Then, since it is an even-parity polynomial, it can have an even number of zeros, and since it is of $2n$-degree, it can have up to $2n$ zeros. Thus, $p_{2n}(x)$ can have $0,2,\ldots,2n$ zeros, which are also the zeros of the wave function (1), which thus can be the ground-state wave function, or the second-excited-state wave function,…, or the $2n$-th-excited-state wave function of the oscillator (12). Therefore, the $n+1$ eigenstates we find are, respectively, the ground state, the second-excited state,…, the $2n$-th-excited state of the sextic anharmonic oscillator (12).

To conclude this section, we'll quasi-exactly solve the oscillator (12) for $n=0,1$. For $n=0$, we calculate the ground-state energy and wave function of the sextic anharmonic oscillator

[1] See ref. [10] for some interesting exceptions.

$$V(x) = x^6 - 2g_2x^4 + (g_2{}^2 - 3)x^2$$

In this case, the energy equation is, from (19),

$$f_1(q_0(0)) = 0,$$

and since $f_1(q_0(0)) = -q_0(0)/2$, we obtain

$$q_0(0) = 0$$

Then, from (10), the ground-state energy is

$$E_0 = -g_2$$

For $n = 0$, $p_0(x) = 1$ and from (1), the ground-state wave function is

$$\psi_0(x) = A_0 \exp\left(-\frac{1}{4}x^4 + \frac{g_2}{2}x^2\right)$$

For $n = 1$, we calculate the ground-state and the second-excited-state energies and wave functions of the sextic anharmonic oscillator

$$V(x) = x^6 - 2g_2x^4 + (g_2{}^2 - 7)x^2$$

In this case, the energy equation is, from (18),

$$q_0(2) f_1(q_0(2)) + 4g_2 f_1(q_0(2)) + 4f_0(q_0(2)) = 0,$$

and using that $f_0(q_0(2)) = 1$ and $f_1(q_0(2)) = -q_0(2)/2$, we end up to the equation

$$q_0{}^2(2) + 4g_2q_0(2) - 8 = 0,$$

which has two real roots, namely

$$q_0(2) = -2g_2 \pm 2\sqrt{g_2{}^2 + 2}$$

Then, from (10), we obtain the ground-state energy and the second-excited-state energy, which are, respectively,

$$E_0 = -3g_2 - 2\sqrt{g_2{}^2 + 2}$$

$$E_2 = -3g_2 + 2\sqrt{g_2{}^2 + 2}$$

The polynomial $p_2(x)$ has the form

$$p_2(x) = x^2 + p_0$$

To calculate p_0, we use (16), which for $k = n = 1$ is written as

$$f_1\left(q_0\left(2\right)\right)p_0=1$$

Then, since $f_1\left(q_0\left(2\right)\right)=-q_0\left(2\right)/2$, we obtain for the two values of $q_0\left(2\right)$, respectively,

$$p_0=\frac{1}{g_2-\sqrt{{g_2}^2+2}} \text{ and } p_0=\frac{1}{g_2+\sqrt{{g_2}^2+2}}$$

The first one corresponds to the value of $q_0\left(2\right)$ with the plus sign, i.e. it corresponds to the second-excited state energy, while the second one corresponds to the ground-state energy. Then, using (1), the ground-state wave function is

$$\psi_0\left(x\right)=A_0\left(x^2+\frac{1}{g_2+\sqrt{{g_2}^2+2}}\right)\exp\left(-\frac{1}{4}x^4+\frac{g_2}{2}x^2\right)$$

and the second-excited-state wave function is

$$\psi_2\left(x\right)=A_2\left(x^2+\frac{1}{g_2-\sqrt{{g_2}^2+2}}\right)\exp\left(-\frac{1}{4}x^4+\frac{g_2}{2}x^2\right)$$

Observe that the ground-state wave function has no zeros, while the second-excited-state wave function has two zeros[2], as expected.

4. The odd-parity case

This case is similar to the even-parity case and we'll examine it in brief.

Since the polynomial $p_n\left(x\right)$ is of n degree, n must be odd in this case, and from (1) we see that the ansatz eigenfunction is also of odd parity. Setting $n\to 2n+1$, with $n=0,1,\ldots$, the oscillator (11) takes the form

$$V\left(x\right)=x^6-2g_2x^4+\left({g_2}^2-\left(3+2\left(2n+1\right)\right)\right)x^2 \tag{22}$$

while the differential equation (8) becomes

$$p_{2n+1}''\left(x\right)+2\left(-x^3+g_2x\right)p_{2n+1}'\left(x\right)=-\left(2\left(2n+1\right)x^2+q_0\left(2n+1\right)\right)p_{2n+1}\left(x\right) \tag{23}$$

where

$$p_{2n+1}\left(x\right)=x^{2n+1}+p_{2n-1}x^{2n-1}+\ldots+p_1x$$

[2] Since $\sqrt{{g_2}^2+2}>\left|g_2\right|=\pm g_2$, we obtain $g_2+\sqrt{{g_2}^2+2}>0$ (from the second inequality) and $g_2-\sqrt{{g_2}^2+2}<0$ (from the first inequality).

In (23), we have only odd-degree terms in x, and equating the coefficients of the $(2k+1)$-degree terms in x on both sides of (23) yields

$$(2k+3)(2k+2)\,p_{2k+3} = -\left(2(2k+1)\,g_2 + q_0\,(2n+1)\right) p_{2k+1} - 4(n+1-k)\,p_{2k-1} \tag{24}$$

As in the even-parity case, the recursion relation (24) holds non-trivially for $k = 0,1,...,n$.

Using the same reasoning as in the even-parity case, we derive the following relation

$$p_{2k+1} = f_k\left(q_0\,(2n+1)\right) p_1 \tag{25}$$

for $k = 0,1,...n$, with $f_k\left(q_0\,(2n+1)\right)$ a polynomial of degree k in $q_0(2n+1)$, and $f_0\left(q_0\,(2n+1)\right) = 1$ and $f_1\left(q_0\,(2n+1)\right) = -\left(q_0\,(2n+1) + 2g_2\right)/6$.

Using (25), the recursion relation (24) is written in terms of $f_k\left(q_0\,(2n+1)\right)$ as

$$\begin{aligned}&(2k+3)(2k+2)\,f_{k+1}\left(q_0\,(2n+1)\right) = \\ &= -\left(2(2k+1)\,g_2 + q_0\,(2n+1)\right) f_k\left(q_0\,(2n+1)\right) - 4(n+1-k)\,f_{k-1}\left(q_0\,(2n+1)\right)\end{aligned} \tag{26}$$

for $k = 0,1,...,n$. As in the even-parity case, for $k = n$ we obtain the energy equation

$$q_0\,(2n+1)\,f_n\left(q_0\,(2n+1)\right) + 2(2n+1)\,g_2 f_n\left(q_0\,(2n+1)\right) + 4 f_{n-1}\left(q_0\,(2n+1)\right) = 0 \tag{27}$$

or, equivalently,

$$f_{n+1}\left(q_0\,(2n+1)\right) = 0 \tag{28}$$

In the same way as in the even-parity case, we can show that the polynomial $f_k\left(q_0\,(2n+1)\right)$ has k (real) zeros, and thus from (28) we obtain $n+1$ different values of $q_0(2n+1)$, and then from (10) we take $n+1$ different energies, which are non-degenerate and correspond to $n+1$ eigenstates of the sextic anharmonic oscillator (22).

As seen from (25), $p_1 \neq 0$, otherwise all coefficients of $p_{2n+1}(x)$ vanish, which is impossible, as $p_{2n+1}(x)$ is monic. Then, $p_{2n+1}(x) = xQ_{2n}(x)$, where $Q_{2n}(x)$ an even-parity, $2n$-degree polynomial, which does not vanish at zero, as $Q_{2n}(0) = p_1 \neq 0$, and thus it has an even number of zeros. Then, the polynomial $p_{2n+1}(x)$ has an odd number of zeros, and since it is of $(2n+1)$-degree, it can have up to $2n+1$ zeros, i.e. it can have $1,3,\ldots,2n+1$ zeros, which are also the zeros of the wave function (1), which thus can be, according to the node theorem [9], the first-excited-state wave function, or the third-excited-state wave function,…, or the $(2n+1)$-th-excited-state wave function of the oscillator (22). Therefore, the $n+1$ eigenstates we find are, respectively, the first-excited state, the third-excited state,..., the $(2n+1)$-th-excited state of the sextic anharmonic oscillator (22).

We'll conclude this section by quasi-exactly solving the oscillator (22) for $n=0,1$.

For $n=0$, we calculate the first-excited-state energy and wave function of the sextic anharmonic oscillator

$$V(x)=x^6-2g_2x^4+\left(g_2{}^2-5\right)x^2$$

In this case, the energy equation is, from (28),

$$f_1\left(q_0(1)\right)=0,$$

and using that $f_1\left(q_0(1)\right)=-\left(q_0(1)+2g_2\right)/6$, we obtain

$$q_0(1)=-2g_2$$

Then, from (10), the first-excited-state energy is

$$E_1=-3g_2$$

For $n=0$, $p_1(x)=x$ and from (1), the first-excited-state wave function is

$$\psi_1(x)=A_1x\exp\left(-\frac{1}{4}x^4+\frac{g_2}{2}x^2\right)$$

For $n=1$, we calculate the first and third-excited-state energies and wave functions of the sextic anharmonic oscillator

$$V(x)=x^6-2g_2x^4+\left(g_2{}^2-9\right)x^2$$

In this case, the energy equation is, from (27),

$$q_0(3)f_1\left(q_0(3)\right)+6g_2f_1\left(q_0(3)\right)+4f_0\left(q_0(3)\right)=0,$$

and using that $f_0\left(q_0(3)\right)=1$ and $f_1\left(q_0(3)\right)=-\left(q_0(3)+2g_2\right)/6$, we end up to the equation

$$q_0{}^2(3)+8g_2q_0(3)+12g_2{}^2-24=0,$$

which has two real roots, namely

$$q_0(3)=-4g_2\pm2\sqrt{g_2{}^2+6}$$

Then, from (10), we obtain the first and third-excited-state energies, which are, respectively,

$$E_1=-5g_2-2\sqrt{g_2{}^2+6}$$

$$E_3=-5g_2+2\sqrt{g_2{}^2+6}$$

The polynomial $p_3(x)$ has the form

$$p_3(x) = x^3 + p_1 x$$

To calculate p_1, we use (25), which for $k = n = 1$ is written as

$$f_1(q_0(3))\, p_1 = 1$$

Then, since $f_1(q_0(3)) = -(q_0(3) + 2g_2)/6$, we obtain for the two values of $q_0(3)$, respectively,

$$p_1 = \frac{3}{g_2 - \sqrt{g_2^{\,2} + 6}} \text{ and } p_1 = \frac{3}{g_2 + \sqrt{g_2^{\,2} + 6}}$$

The first one corresponds to the value of $q_0(3)$ with the plus sign, i.e. it corresponds to the third-excited-state energy, while the second one corresponds to the first-excited-state energy. Then, using (1), the first-excited-state wave function is

$$\psi_1(x) = A_1 x \left(x^2 + \frac{3}{g_2 + \sqrt{g_2^{\,2} + 6}} \right) \exp\left(-\frac{1}{4}x^4 + \frac{g_2}{2}x^2 \right)$$

and the third-excited-state wave function is

$$\psi_3(x) = A_3 x \left(x^2 + \frac{3}{g_2 - \sqrt{g_2^{\,2} + 6}} \right) \exp\left(-\frac{1}{4}x^4 + \frac{g_2}{2}x^2 \right)$$

Observe that the first-excited-state wave function has one zero, while the third-excited-state wave function has three zeros, as expected.

5. Energy reflection symmetry if the coupling constant vanishes

The coupling constant of the sextic anharmonic oscillator (11) is $-2g_2$, and thus it vanishes if (and only if) g_2 vanishes, and then the oscillator becomes

$$V(x) = x^6 - (3 + 2n)x^2 \tag{29}$$

The oscillator (29) has energy reflection symmetry, i.e. its known non-zero energies (10) come in symmetric pairs $(E, -E)$.

Proof

If g_2 vanishes, the recursion relations (17) and (26), for the polynomial f_k in the even and odd-parity cases, become, respectively,

$(2k+2)(2k+1)f_{k+1}(q_0(2n)) = -q_0(2n)f_k(q_0(2n)) - 4(n+1-k)f_{k-1}(q_0(2n))$,

with $f_0(q_0(2n)) = 1$ and $f_1(q_0(2n)) = -q_0(2n)/2$, and

$(2k+3)(2k+2)f_{k+1}(q_0(2n+1)) = -q_0(2n+1)f_k(q_0(2n+1)) - 4(n+1-k)f_{k-1}(q_0(2n+1))$,

with $f_0(q_0(2n+1)) = 1$ and $f_1(q_0(2n+1)) = -q_0(2n+1)/6$. In both cases, f_0 is of even parity and f_1 is of odd parity, and if f_{k-1} and f_k have different parities, f_{k+1} has the same parity as f_{k-1}. Thus, in both cases, f_k is of even/odd parity if k is even/odd. Then, the polynomial f_{n+1} in (19) and (28) has definite parity, and thus if q_0 is a root, $-q_0$ is also a root. Then, since in this case q_0 is equal to the energy (10), if E is an energy, $-E$ is also an energy, and thus the known non-zero energies (10) of the sextic anharmonic oscillator (29) come in symmetric pairs.

6. Concluding remarks

Using as basic parameter, the constant term of a quotient polynomial, we have shown that the sextic anharmonic oscillator (11) is quasi-exactly solvable, in the sense that for every value of the non-negative integer n, $\left\lfloor \frac{n}{2}+1 \right\rfloor$ (see foot note [3]) energies and the respective eigen functions can be exactly calculated by the relation (10) and the ansatz (1), respectively.

The method we have presented can be straightforwardly used to study not only real and analytic polynomial potentials, but also piecewise-analytic or complex PT-symmetric polynomial potentials, such as the symmetrized quartic and sextic anharmonic oscillators that have been introduced in [11, 12], or the complex PT-symmetric quartic and sextic anharmonic oscillators that have been studied in [13, 14]. In the cases of symmetrized or complex PT-symmetric sextic oscillators, the quotient polynomial has, respectively, a non-analytic linear term or a linear term with an imaginary coefficient and if this term vanishes, the quotient polynomial becomes analytic or real, respectively. Then, the use of our approach opens up the possibility to search for symmetrized (non-analytic) or complex PT-symmetric sextic oscillators resulting, respectively, from analytic or real quotient polynomials, and to identify a possible dynamical symmetry which distinguishes those oscillators from the other of the same class.

[3] The symbol $\lfloor\ \rfloor$ denotes the floor function, i.e. $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

References

[1] A. G. Ushveridze, *Quasi-Exactly Solvable Models in Quantum Mechanics*, (IOPP, Bristol, 1994).

[2] A. V. Turbiner, "Quasi-Exactly-Solvable Problems and sl(2) Algebra", *Commun. Math. Phys.* **118**, 467-474 (1988).

[3] Alexander V. Turbiner, "One-dimensional quasi-exactly solvable Schrödinger equations", *Phys. Rep.* **642** 1-71 (2016).

[4] Carl M. Bender and Gerald V. Dunne, "Quasi-Exactly Solvable Systems and Orthogonal Polynomials", *J. Math. Phys.* **37** 6-11 (1996).

[5] Yao-Zhong Zhang, "Exact polynomial solutions of second order differential equations and their applications", *J. Phys. A: Math. Theor.* **45**, 065206 (2012).

[6] Choon-Lin Ho, "Prepotential approach to exact and quasi-exact solvabilities of Hermitian and non-Hermitian Hamiltonians", *Ann. Phys.* **323**: 2241-2252 (2008).

[7] Hakan Ciftci, Richard L Hall, Nasser Saad, and Ebubekir Dogu, "Physical applications of second-order linear differential equations that admit polynomial solutions*", J. Phys. A: Math. Theor.* **43**, 415206 (2010).

[8] David J. Griffiths, *Introduction to Quantum Mechanics*, (Pearson Education, Inc., Second Edition, 2005).

[9] M. Moriconi, "Nodes of wavefunctions", *Am. J. Phys.* **75**, 284-285 (2007).

[10] Sayan Kar and Rajesh R. Parwani, "Can degenerate bound states occur in one dimensional quantum mechanics?", *EPL (Europhysics Letters)* **80** (3), 30004 (2007).

[11] Miloslav Znojil, "Symmetrized quartic polynomial oscillators and their partial exact solvability", *Phys. Lett. A* **380**, 1414-1418 (2016).

[12] C. Quesne, "Quasi-exactly solvable symmetrized quartic and sextic polynomial oscillators", *Eur. Phys. J. Plus* **132**: 450 (2017).

[13] Carl M. Bender and Stefan Boettcher, "Quasi-exactly solvable quartic potential", *J. Phys. A: Math. Gen.* **31** L273 (1998).

[14] B. Bagchi, F. Cannata, and C. Quesne, "PT-symmetric sextic potentials", *Phys. Lett. A* **269** 79-82 (2000)